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Evaluation of minors associated to weighing matrices

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Abstract

In the present paper we focus our research on calculating minors of weighing matrices of order n and weight $n-k$, denoted by $W(n,n-k)$. We provide analytical determinant computations, counting techniques for specifying the existence of certain submatrices inside a $W(n,n-k)$ and an algorithm for computing the $(n-j)\times(n-j)$ minors of a $W(n,n-k)$, which is realized with the notion of symbolic manipulation. These results are valid of general n. The ideas presented in this work can be used as the fundamental basis, on which the calculation of minors of other weighing matrices, and in general of orthogonal matrices, can be developed. An application of the derived formulas to an interesting problem of Numerical Analysis, the growth problem, is also presented.

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1. Introduction

Determinants have a long history in mathematics and arise in numerous applications, mostly as tools for solving linear systems of equations, matrix inversion and eigenvalue problems. As a consequence, they have been researched extensively, which has yielded efficient algorithms for determinant computation of several matrix classes. Determinants are nowadays still

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a very challenging issue, e.g. see [7], [1] and [12]. Generally, it is very difficult to derive analytical formulas for the determinant of a given matrix, or for the minors of it. Laplace's famous Minor Expansion Theorem provides an important method to recursively compute the value of a determinant, but it usually requires a high computational cost. Therefore, when we have matrices of special structure, it is challenging to determine analytical formulas, if possible. This happened already for Hadamard matrices, Vandermonde matrices and Hankel matrices.

So, why do we want to have determinant formulas for specially structured matrices when we have methods to compute every general determinant? Consider det V_3, where

$$V_3 = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}.$$

Using minor expansion we obtain det $V_3 = x_2x_3^2 - x_2^2x_3 - x_1x_3^2 + x_1^2x_3 + x_1x_2^2 - x_1^2x_2$. Surprisingly, even a computer algebra system like Maple that provides efficient symbolic computation fails to compute the determinant of an equally structured matrix (i.e. $v_{ij} = x_i^{j-1}$) of dimension 8, since the size of the intermediate results cannot be handled. However, the determinant of V_n, which is known as *Vandermonde matrix*, has the simple formula

$$\det V_n = \prod_{1 \leqslant i \leqslant j \leqslant n} (x_j - x_i)$$

If we blindly expand the determinant then we also often lose the structure of the matrix, like it is not immediately obvious from the first expression for det V_3 that det $V_3 = 0$ for $x_i = x_j$, whereas the representation det $V_3 = (x_3 - x_2)(x_2 - x_1)(x_3 - x_1)$, according to the second expression, gives immediate insight, offering a clearly simpler structure.

In our research we are interested in calculating the minors of weighing matrices $W(n, n-k)$ for n even and $k \geqslant 1$. A $(0, 1, -1)$ matrix $W = W(n, n-k)$, $k = 1, 2, \ldots$, of order n satisfying $W^{\mathrm{T}}W = WW^{\mathrm{T}} = (n-k)I_n$ is called a *weighing matrix of order n and weight n − k* or simply a *weighing matrix*. A $W(n, n)$, $n \equiv 0 \pmod 4$, is a *Hadamard matrix of order n*. A $W = W(n, n-k)$ for which $W^{\mathrm{T}} = -W$, $n \equiv 0 \pmod 4$, is called a *skew–weighing matrix*. A $W = W(n, n-1)$ satisfying $W^{\mathrm{T}} = W$, $n \equiv 2 \pmod 4$, is called a *symmetric conference matrix*. For more details on weighing matrices the reader can consult [5]. Two important properties of the weighing matrices, which follow directly from the definition, are:

1. Every row and column of a $W(n, n-k)$ contains exactly k zeros;
2. Every two distinct rows and columns of a $W(n, n-k)$ are orthogonal to each other, which means that their inner product is zero.

Two matrices are said to be *Hadamard equivalent* or *H-equivalent* if one can be obtained from the other by a sequence of the operations:

1. interchange any pairs of rows and/or columns;
2. multiply any rows and/or columns through by -1.

The need for studying $W(n, n-k)$ rises from their interesting properties during Gaussian Elimination (GE) [10] on Completely Pivoted (CP, no row and column exchanges are needed during GE with complete pivoting) weighing matrices, and also from their application in several areas of Applied Mathematics, such as Theory of Experimental Designs [6] and Coding Theory [4].

The paper is organized as follows. In Section 2 we prove analytical formulas for minors of $W(n, n-k)$. In Section 3 we develop counting techniques, which can be used for specifying the existence or non-existence of any matrices inside a $W(n, n-k)$, and moreover for the specification of values of minors of $W(n, n-k)$. In Section 4 we present an algorithm for computing the $(n-j) \times (n-j)$ minors of a $W(n, n-k)$, which is realized with the notion of symbolic manipulation. The above described techniques can be adopted for other orthogonal matrices, too, after appropriate modifications. For brevity, however, we will not discuss such variations here. In Section 5 we describe a very challenging problem of Numerical Analysis, the growth problem, and show how the previous results can be related to it. The motivation for our research is the fact that the pivots appearing after GE on a CP matrix are strictly connected with minors of the matrix (relation (19)). So, we can take advantage of the computed formulas and use them for specifying desired pivot patterns. The numerical experiments presented in Section 6 indicate interesting properties of the weighing matrices in regard to the growth problem.

Notations. Throughout this paper we assume that the order n of all appearing $W(n, n-k)$ is even. We also assume, without loss of generality, that the first non-zero entry of a row and a column of a weighing matrix is always $+1$, because this can be achieved with the H-equivalent operation of multiplying by -1 and leaves unaffected the magnitude of the determinant, in which we are actually interested. The elements of a $(0, 1, -1)$ matrix will be denoted by $(0, +, -)$. I_n and J_n stand for the identity matrix of order n and the matrix with ones of order n, respectively. We write $W(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper left corner (ULC) of the matrix W. We denote with $x_{m\times n}$ the $m \times n$ block with elements x, x real, and with $X_{m\times n}$ the $m \times n$ block with the specific form of the matrix X.

Let $\underline{x}^{\mathrm{T}}_{\beta+1}$ the vectors containing the binary representation of each integer $\beta + 2^{j-1}$ for $\beta = 0, \ldots, 2^{j-1} - 1$. Replace all zero entries of $\underline{x}^{\mathrm{T}}_{\beta+1}$ by -1 and define the $j \times 1$ vectors $\underline{u}_k = \underline{x}_{2^{j-1}-k+1}$, $k = 1, \ldots, 2^{j-1}$. We write U_j for all the matrices with j rows and the appropriate number of columns, in which $\underline{u}_k$ occurs u_k times. So

$$U_j = \begin{array}{cccc} \overbrace{+\cdots+}^{u_1} & \overbrace{+\cdots+}^{u_2} & \cdots\ \overbrace{+\cdots+}^{u_{2^{j-1}-1}} & \overbrace{+\cdots+}^{u_{2^{j-1}}} \\ +\cdots+ & +\cdots+ & \cdots\ -\cdots- & -\cdots- \\ \cdot & \cdot & \cdots\quad \cdot & \cdot \\ \cdot & \cdot & \cdots\quad \cdot & \cdot \\ +\cdots+ & +\cdots+ & \cdots\ -\cdots- & -\cdots- \\ +\cdots+ & -\cdots- & \cdots\ +\cdots+ & -\cdots- \end{array} = \begin{array}{ccccc} u_1 & u_2 & \cdots & u_{2^{j-1}-1} & u_{2^{j-1}} \\ + & + & \cdots & + & + \\ + & + & \cdots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ + & + & \cdots & - & - \\ + & - & \cdots & + & - \end{array}$$

Example 1. $U_3 = \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ + & + & + & + \\ + & + & - & - \\ + & - & + & - \end{array}$ and $U_4 = \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{array}$

2. Analytical formulas for minors of $W(n, n-k)$

In the present section we present techniques for the yield of analytical formulas associated with minors of weighing matrices. The whole approach with appropriate modifications can be also applied on Hadamard matrices, which are a special case of $W(n, n-k)$ for $k = 0$ and $n \equiv 0 \pmod 4$. The application of these ideas on other classes of orthogonal matrices is an issue currently under investigation.

Preliminary Results. 1. Let $A = (k-\lambda)I_v + \lambda J_v$, where k, λ are integers. Then,

$$\det A = [k + (v-1)\lambda](k-\lambda)^{v-1} \tag{1}$$

and

$$A^{-1} = \frac{1}{k^2 + (v-2)k\lambda - (v-1)\lambda^2}\{[k + (v-2)\lambda + \lambda]I - \lambda J\}. \tag{2}$$

2. Let $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$. Then

$$\det B = \det B_1 \cdot \det(B_4 - B_3 B_1^{-1} B_2). \tag{3}$$

For a weighing matrix $W(n, n-k)$, since $WW^{\mathrm{T}} = (n-k)I_n$, we have that

$$\det W \equiv W(n) = (n-k)^{\frac{n}{2}} \tag{4}$$

Next we proceed to evaluating analytically minors of $W(n, n-k)$ matrices, according to the following technique.

Evaluation technique. The proposed strategy is described in a general context as follows. In order to standardize a technique for calculating all possible $n-r$ minors of a $W = W(n, n-k)$, $r = 1, 2, 3$, we assume, without loss of generality, a pattern for the first r rows of every $W(n, n-k)$, which will be proved to exist always, and we single out from this pattern r columns and write them separately in the upper left corner. This is done indeed without loss of generality because, if the first r rows do not appear in the suggested form, we can make this form to appear by performing appropriate row and/or column interchanges and/or column multiplications by -1. These operations do not affect the determinant of the matrix, and moreover the values of the $n-r$ minors, and aim at the grouping (clustering) of same columns. In this manner, the computations done by exploiting the orthogonality relation $W^{\mathrm{T}}W = (n-k)I_n$ are facilitated due to the block forms of the appearing matrices and eventually they lead to general theoretical formulas. On the other hand, the fact that *for every possible upper left $r \times r$ corner* we calculate the determinant of the lower right $(n-r) \times (n-r)$ submatrix, which is actually the desired $n-r$ minor, guarantees that with this technique we calculate *every* possible $n-r$ minor of W and that we do not miss out any appearing values. Finally, it is important to stress that we have chosen to single out the first r rows and columns without any loss of generality, only for the sake of better presentation and for standardizing a convenient technique. These r rows and columns can be actually located *everywhere* inside the matrix. The above ideas can be understood better through the material in the rest of the section.

Proposition 1. *Let W be a $W(n, n-k)$, $k \geqslant 1$. Then all possible $(n-1) \times (n-1)$ minors of W are*: 0 *and* $(n-k)^{\frac{n}{2}-1}$.

Proof. Since W is a $W(n, n-k)$ we suppose that it can be written in one of the following forms:

$$W = \left[\begin{array}{c|cccc} + & \overbrace{0\,0\cdots 0}^{k} & + & \cdots & + \\ \hline 0 & & & & \\ \vdots & & & & \\ 0 & & B & & \\ + & & & & \\ \vdots & & & & \\ + & & & & \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|cccc} 0 & \overbrace{0\,0\cdots 0}^{k-1} & + & \cdots & + \\ \hline 0 & & & & \\ \vdots & & & & \\ 0 & & B' & & \\ + & & & & \\ \vdots & & & & \\ + & & & & \end{array}\right],$$

where the first columns contain also k and $k-1$ 0's below the horizontal line, respectively.

From the definition of the $W(n, n-k)$, $WW^{\mathrm{T}} = (n-k)I_n$, it follows that the $(n-1)\times(n-1)$ matrix BB^{T} has the form

$$BB^{\mathrm{T}} = \begin{bmatrix} C & \mathbf{O} \\ \mathbf{O} & D \end{bmatrix},$$

where $C = (n-k)I_k$ and $D = (n-k)I_{n-k-1} - J_{n-k-1}$. Obviously the blocks with 0's in the upper right and lower left corner of BB^{T} are of order $k \times (n-k-1)$ and $(n-k-1)\times k$, respectively.

Then, from (1), we have
$\det BB^{\mathrm{T}} = \det C \cdot \det D = (n-k)^k[n-k-1-(n-k-2)](n-k)^{n-k-2} = (n-k)^{n-2}$.

So $\det B = (n-k)^{\frac{n}{2}-1}$.

Working similarly for the second form of a $W(n, n-k)$ yields $\det B' = 0$. $\square$

Lemma 1. *If we fix one specific row of a $W(n, n-k)$, $k \geqslant 1$, we can always find a second row, so that the two rows have the form*:

$$\begin{array}{ccccc} \overbrace{0\,0\cdots 0}^{j} & \overbrace{0\cdots 0}^{k-j} & \overbrace{+\cdots +}^{k-j} & \overbrace{+\cdots +}^{s} & \overbrace{+\cdots +}^{s} \\ 0\,0\cdots 0 & +\cdots + & 0\cdots 0 & +\cdots + & -\cdots - \end{array}, \qquad (P)$$

for some j even, $0 \leqslant j \leqslant k$. This can be always achieved by performing the appropriate H-equivalent operations. Particularly for $k = 1$, the result holds trivially for $j = 0$.

Proof. Suppose that the pattern (P) cannot exist for any j, $2 \leqslant j \leqslant k$, $k \geqslant 2$. Then, obviously, two rows of a $W(n, n-k)$ either have no common zeros (which corresponds to the case $j = 0$) or they can be written, up to H-equivalence, as

$$\begin{array}{cccc} \overbrace{0\,0\cdots 0}^{k} & \overbrace{+\cdots +}^{k-1} & \overbrace{+\cdots +}^{s_1} & \overbrace{+\cdots +}^{s_2} \\ 0+\cdots + & 0\cdots 0 & +\cdots + & -\cdots - \end{array}$$

Indeed, this can be always done by performing column interchanges and column multiplications by -1, if necessary. From the inner product of these two rows we get $s_1 - s_2 = 0 \Rightarrow s_1 = s_2 \equiv s$.

From the order of the matrix we have: $2k - 1 + 2s = n \Rightarrow 2k + 2s - n = 1$.

Since n is even, the left side of the previous equality is an even number, while the right side is odd. This is a contradiction, so the validity of the result is proved. Hence, it follows that two rows of a $W(n, n-k)$ will have the form

$$\begin{array}{ccccc} \overbrace{0\,0\cdots 0}^{j} & \overbrace{0\cdots 0}^{k-j} & \overbrace{+\cdots+}^{k-j} & \overbrace{+\cdots+}^{s} & \overbrace{+\cdots+}^{s}, \\ 0\,0\cdots 0 & +\cdots+ & 0\cdots 0 & +\cdots+ & -\cdots- \end{array}$$

for some $j \geqslant 2$. Obviously, $j \leqslant k$ because j cannot exceed the number of zeros per row. It remains to show that j is even. Considering again the order of the matrix we have: $2k - j + 2s = n \Rightarrow 2k + 2s - n = j$.

The left side of the previous equality is an even number, hence j must be even.

The second part of the enunciation of the lemma is trivial.

For $k = 1$ it is easy to see that two rows of a $W(n, n-1)$ can have the form

$$\begin{array}{cccc} 0 & + & \overbrace{+\cdots+}^{s} & \overbrace{+\cdots+}^{s}, \\ + & 0 & +\cdots+ & -\cdots- \end{array}$$

so the result is valid trivially for $j = 0$. □

Remark 1. Since the above pattern (P) always exists among the rows of a $W(n, n-k)$, without loss of generality we assume that it appears in the first two rows of any $W(n, n-k)$. In the rest of the section we will consider any $W(n, n-k)$ in this form.

Corollary 1. *If we fix one specific row of a $W(n, n-2)$, we can always find a second row so that the two rows have the form*

$$\begin{array}{cccc} 0 & 0 & \overbrace{+\cdots+}^{s} & \overbrace{+\cdots+}^{s}. \\ 0 & 0 & +\cdots+ & -\cdots- \end{array}$$

This can be always achieved by performing the appropriate H-equivalent operations.

Proof. From Lemma 1 we have that there always exists the $2 \times j$ block with zeros for some j even. Since $k = 2$, from the definition of a $W(n, n-2)$ we have that every row must contain exactly 2 zeros, so necessarily $j = 2$ and the $k - j$ columns $[0\ +]^{\mathrm{T}}$ and $[+\ 0]^{\mathrm{T}}$ of (P) vanish in this case. □

Remark 2. In accordance with Lemma 1, the following form of two rows of a $W(n, n-2)$

$$\begin{array}{cccccc} 0 & 0 & + & + & \overbrace{+\cdots+}^{s} & \overbrace{+\cdots+}^{s}, \\ + & + & 0 & 0 & +\cdots+ & -\cdots- \end{array}$$

which corresponds to $j = 0$, is also possible to appear, as it can be seen in the third matrix of Example 2. However, for the sake of better presentation and convenience, we choose the form given in Corollary 1 in order to standardize a pattern for two rows of a $W(n, n-2)$.

Example 2. The results of Lemma 1 and Corollary 1 can be verified in the following weighing matrices $W(n, n-k)$

$$W(8,4)=\begin{bmatrix} + & + & 0 & 0 & 0 & 0 & + & + \\ + & - & 0 & 0 & + & + & 0 & 0 \\ 0 & 0 & - & + & 0 & 0 & - & + \\ 0 & 0 & - & - & - & + & 0 & 0 \\ 0 & + & + & 0 & 0 & + & - & 0 \\ 0 & + & - & 0 & + & 0 & 0 & - \\ + & 0 & 0 & + & - & 0 & 0 & - \\ + & 0 & 0 & - & 0 & - & - & 0 \end{bmatrix}$$

$$W(12,8)=\begin{bmatrix} + & + & 0 & 0 & 0 & 0 & + & + & + & + & + & + \\ + & - & 0 & 0 & + & + & 0 & 0 & + & + & - & - \\ 0 & 0 & - & - & + & - & 0 & 0 & + & - & - & + \\ 0 & 0 & - & + & 0 & 0 & + & - & + & - & + & - \\ 0 & + & - & 0 & 0 & - & + & 0 & - & + & - & - \\ 0 & + & + & 0 & - & 0 & 0 & + & + & - & - & - \\ + & 0 & 0 & + & 0 & + & + & 0 & - & - & - & + \\ + & 0 & 0 & - & + & 0 & 0 & + & - & - & + & - \\ + & + & - & - & - & + & - & - & 0 & 0 & 0 & 0 \\ + & + & + & + & + & - & - & - & 0 & 0 & 0 & 0 \\ + & - & + & - & - & - & + & - & 0 & 0 & 0 & 0 \\ + & - & - & + & - & - & - & + & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the above $W(8,4)$ we see that for any row there exists always a second one, so that these two rows satisfy Lemma 1 with $j=2$, after the appropriate H-equivalent operations are performed, e.g. the first row together with the second row, or alternatively the third together with the fourth row etc. In the $W(12,8)$ we observe that any choice of pair of rows among its four last rows satisfies Lemma 1 with $j=4$, hence, since $j=k=4$ in this case, the $k-j$ columns $[0\,+]^{\mathrm{T}}$ and $[+\,0]^{\mathrm{T}}$ of (P) vanish. All other rows of it satisfy Lemma 1 with $j=2$, always after selecting the appropriate pair

$$W(8,6)=\begin{bmatrix} - & - & 0 & - & 0 & - & + & - \\ - & - & - & 0 & - & 0 & - & + \\ 0 & + & - & - & + & - & 0 & + \\ + & 0 & - & - & - & + & + & 0 \\ 0 & + & - & + & - & - & 0 & - \\ + & 0 & + & - & - & - & - & 0 \\ - & + & 0 & - & 0 & + & - & - \\ + & - & - & 0 & + & 0 & - & - \end{bmatrix}.$$

We see the 2×2 block with zeros described by Corollary 1 inside this $W(8,6)$ e.g. in rows 1 and 7, or 2 and 8, or 3 and 5 etc. If we make some column interchanges and multiplications of columns by -1, we can have these pairs of rows in the form of Corollary 1. If we make at most two additional row interchanges, we can have these two rows without loss of generality as the first two rows of the matrix. The rows of this matrix satisfy Lemma 1 also for $j=0$ as it explained in Remark 2. This can be seen if, for instance, one selects the first row together with the second, or the third row together with the fourth, or the first row with the third etc. We observe that if we fix a specific row R_1, there can be found $n-2$ second rows R_2 so that R_1 and R_2 satisfy Lemma 1 for $j=0$, while there is only one row R_2 so that the Lemma is satisfied for $j=2$.

Proposition 2. *Let W be a $W(n, n-k)$, $k \geqslant 1$. Then all possible $(n-2) \times (n-2)$ minors of W are*: 0, $(n-k)^{\frac{n}{2}-2}$ *and* $2(n-k)^{\frac{n}{2}-2}$.

Proof. Let

$$W = \left[\begin{array}{ccccc} \overbrace{0\,0\cdots 0}^{j} & \overbrace{0\cdots 0}^{k-j} & \overbrace{+\cdots+}^{k-j} & \overbrace{+\cdots+}^{s} & \overbrace{+\cdots+}^{s} \\ 0\,0\cdots 0 & +\cdots+ & 0\ldots 0 & +\cdots+ & -\cdots- \\ \hline & & & & \\ & & & & \\ & & & & \end{array}\right] = \left[\begin{array}{c} (P) \\ \hline \\ \\ \end{array}\right]$$

be a $W(n, n-k)$, $k \geqslant 2$, in the form given in Lemma 1. In order to proceed with the computation for all possible $(n-2) \times (n-2)$ minors we need to specify all possible 2×2 upper left corners that can appear.

There are 10 possible cases, up to H-equivalence, for the upper left 2×2 corner:

$$\begin{bmatrix} + & + \\ + & - \end{bmatrix}, \begin{bmatrix} + & + \\ 0 & \pm \end{bmatrix}, \begin{bmatrix} + & 0 \\ 0 & \pm \end{bmatrix}, \begin{bmatrix} + & + \\ + & + \end{bmatrix},$$
$$\begin{bmatrix} + & + \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & 0 \\ + & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & + \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For every case we single out from pattern (P) the two columns, which make up the corresponding 2×2 matrix, and write them separately in the position of the upper left 2×2 corner. We will carry out the proof for the first case, since the other cases can be handled similarly.

So, in this case, $W = W(n, n-k)$ can be written in the following form, for $k \geqslant 2$:

$$W = \left[\begin{array}{cc|ccccc} + & + & \overbrace{0\,0\cdots 0}^{j} & \overbrace{0\cdots 0}^{k-j} & \overbrace{+\cdots+}^{k-j} & \overbrace{+\cdots+}^{u} & \overbrace{+\cdots+}^{u} \\ + & - & 0\,0\cdots 0 & +\cdots+ & 0\ldots 0 & +\cdots+ & -\cdots- \\ \hline & & & & & & \\ & & & & C & & \\ & & & & & & \end{array}\right],$$

where $u = s - 1$.

From the order of the matrix W we get $2u + 2k - j + 2 = n \Rightarrow u = \frac{n-2k+j-2}{2}$.

According to the definition of the $W(n, n-k)$, $W^{\mathrm{T}}W = (n-k)I_n$, the $(n-2) \times (n-2)$ matrix $C^{\mathrm{T}}C$ has the form

$$C^{\mathrm{T}}C=\begin{bmatrix} C_1 & \mathbf{O}\\ \mathbf{O} & D\end{bmatrix},\quad \text{with } C_1=(n-k)I_j,\quad D=\begin{bmatrix} E & F\\ F^{\mathrm{T}} & G\end{bmatrix},$$

where

$$E=\begin{bmatrix} E_1 & \mathbf{O}\\ \mathbf{O} & E_1\end{bmatrix},\quad E_1=(n-k)I_{k-j}-J_{k-j},$$

$$F=\begin{bmatrix} -1_{(k-j)\times u} & 1_{(k-j)\times u}\\ -1_{(k-j)\times u} & -1_{(k-j)\times u}\end{bmatrix},\quad G=\begin{bmatrix} G_1 & \mathbf{O}\\ \mathbf{O} & G_1\end{bmatrix}\quad \text{and}\quad G_1=(n-k)I_u-2J_u.$$

We have

$$\det C^{\mathrm{T}}C=\det C_1\cdot\det D=(n-k)^j\cdot\det D \tag{5}$$

According to (3),

$$\det D=\det E\cdot\det(G-F^{\mathrm{T}}E^{-1}F). \tag{6}$$

We have $\det E=(\det E_1)^2$ and according to (1),
$\det E_1=[n-k-1-(k-j-1)](n-k-1+1)^{k-j-1}=(n-2k+j)(n-k)^{k-j-1}$, so

$$\det E=(n-2k+j)^2(n-k)^{2(k-j-1)}. \tag{7}$$

After the necessary calculations, with the help of (2), we get

$$X\equiv G-F^{\mathrm{T}}E^{-1}F=\frac{n-k}{n-2k+j}\begin{bmatrix} G_2 & \mathbf{O}\\ \mathbf{O} & G_2\end{bmatrix},$$

where $G_2=(n-2k+j)I_u-2J_u$. Hence,

$$\det X=\left(\frac{n-k}{n-2k+j}\right)^{n-2k+j-2}\cdot(\det G_2)^2. \tag{8}$$

According to (1), $\det G_2=\left[n-2k+j-2-2\left(\frac{n-2k+j-2}{2}-1\right)\right](n-2k+j-2+2)^{\frac{n-2k+j-2}{2}-1}=2(n-2k+j)^{\frac{n-2k+j-4}{2}}$. From (8) follows:

$$\begin{aligned}\det X&=\left(\frac{n-k}{n-2k+j}\right)^{n-2k+j-2}4(n-2k+j)^{n-2k+j-4}\\ &=4(n-k)^{n-2k+j-2}(n-2k+j)^{-2}\end{aligned} \tag{9}$$

Finally, from (5)–(7) and (9) we have

$$\begin{aligned}\det C^{\mathrm{T}}C&=(n-k)^j(n-2k+j)^2(n-k)^{2(k-j-1)}\\ &\quad\times 4(n-k)^{n-2k+j-2}(n-2k+j)^{-2}=4(n-k)^{n-4}.\end{aligned}$$

So, $\det C=2(n-k)^{\frac{n}{2}-2}$.

In a similar manner, the other nine cases give the values 0 and $(n-k)^{\frac{n}{2}-2}$.

For example, for the choice of $\left[\begin{smallmatrix} + & +\\ 0 & +\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0\\ 0 & +\end{smallmatrix}\right]$ as upper left corners, the proof will start with W having as first two rows:

$$\begin{array}{cc|ccccc} + & + & \overbrace{0\,0\,\cdots\,0}^{j} & \overbrace{0\,\cdots\,0}^{k-j} & \overbrace{+\cdots+}^{k-j-1} & \overbrace{+\,\cdots\,+}^{s-1} & \overbrace{+\,\cdots\,+}^{s}\\ 0 & + & 0\,0\,\cdots\,0 & +\cdots+ & 0\cdots 0 & +\,\cdots\,+ & -\,\cdots\,- \end{array}$$

and

$$\begin{array}{cc|ccccc} 0 & 0 & \overbrace{0\,0\cdots 0}^{j-1} & \overbrace{0\cdots 0}^{k-j-1} & \overbrace{+\cdots+}^{k-j} & \overbrace{+\cdots+}^{s} & \overbrace{+\cdots+}^{s}, \\ 0 & + & 0\,0\ldots 0 & +\cdots+ & 0\ldots 0 & +\cdots+ & -\cdots- \end{array}$$

respectively, and following an absolutely similar procedure the rest of the proof is carried out. These cases give $(n-k)^{\frac{n}{2}-2}$ and 0, respectively.

The proof for $k = 1$ is similar and easier. It is done by taking into account that two rows of a $W(n, n-1)$ can have the form given in Lemma 1 and gives as results only the values 0 and $2(n-1)^{\frac{n}{2}-2}$. $\square$

Remark 3. It becomes clear from the above proof that the result of Proposition 2 does not depend on j. This observation is consistent with Lemma 1, which assures the existence of pattern (P) but does not give a specific value for j. So it is intuitively sensible that the result must be independent of j.

The next Proposition 3 computes the $n-3$ minors of a $W(n, n-2)$. Due to the form of the first three rows of a $W(n, n-k)$ we cannot demonstrate the proof generally for an arbitrary weight k, but it can be done similarly to $k = 2$ for every single k fixed. Actually, the problem is that in the general form of a $W(n, n-k)$ there is a number of ± 1's in the second and third row depending on k, which does not allow to discriminate a specific number of cases for the values of these entries. On the contrary, for k fixed we can discriminate specific cases, as it can be seen in the next proof.

Proposition 3. *Let W be a $W(n, n-2)$. Then all possible $(n-3)\times(n-3)$ minors of W are*: 0, $(n-2)^{\frac{n}{2}-3}$, $2(n-2)^{\frac{n}{2}-3}$, $3(n-2)^{\frac{n}{2}-3}$ *and* $4(n-2)^{\frac{n}{2}-3}$.

Proof. Every $W = W(n, n-2)$ can be written in the following form:

$$W = \left[\begin{array}{cccccccc} \text{possible} & & & & & \overbrace{+\cdots+}^{u} & \overbrace{+\cdots+}^{v} & \overbrace{+\cdots+}^{x} & \overbrace{+\cdots+}^{y} \\ 3\times 3 & \underline{c}_1 & \underline{c}_2 & \cdots & \underline{c}_q & +\cdots+ & +\cdots+ & -\cdots- & -\cdots- \\ \text{ULC} & & & & & +\cdots+ & -\cdots- & +\cdots+ & -\cdots- \\ \hline & & & & & & & & \\ & & & & & & & & \end{array}\right],$$

where each column $\underline{c}_i \in \mathbb{R}^{3\times 1}$, $i = 1, \ldots, q$, contains either one or two zeros. The value of q and the position of the zeros in $\underline{c}_i$ vary according to the number and the position of the zeros in the selected 3×3 ULC. The columns $\underline{c}_i$ must be appropriately fixed according to the selected ULC, so that each row of W has exactly two zeros. Next we determine the possible ULC's that can appear. We specified 61 possible cases, up to H-equivalence, for the ULC:

$$\begin{bmatrix} + & + & + \\ + & \pm & \pm \\ + & \pm & \pm \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & \pm & \pm \\ + & \pm & \pm \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & \pm \\ + & \pm & \pm \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & \pm \\ + & \pm & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & + \\ + & + & 0 \\ + & \pm & \pm \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & + \\ + & + & \pm \\ + & \pm & \pm \end{bmatrix},\quad \begin{bmatrix} 0 & 0 & + \\ + & + & 0 \\ + & \pm & 0 \end{bmatrix},\quad \begin{bmatrix} 0 & 0 & + \\ 0 & 0 & + \\ + & + & \pm \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & + \\ 0 & 0 & + \\ + & + & 0 \end{bmatrix}.$$

We will carry out the proof for an ULC of the first case, since the other cases can be handled similarly. Since W is a $W(n, n-2)$, let us suppose

$$W = \left[\begin{array}{ccc|cccccccc} + & + & + & 0 & 0 & + & + & \overbrace{+\cdots+}^{u} & \overbrace{+\cdots+}^{v} & \overbrace{+\cdots+}^{x} & \overbrace{+\cdots+}^{y} \\ + & - & - & 0 & 0 & \pm & \pm & +\cdots+ & +\cdots+ & -\cdots- & -\cdots- \\ + & + & - & + & + & 0 & 0 & +\cdots+ & -\cdots- & +\cdots+ & -\cdots- \\ \hline & & & \multicolumn{8}{c}{} \\ & & & \multicolumn{8}{c}{D} \\ & & & \multicolumn{8}{c}{} \end{array}\right],$$

where $ULC = \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix}$, $\underline{c}_1 = \underline{c}_2 = [0, 0, +]^{\mathrm{T}}$, $\underline{c}_3 = [+, \pm, 0]^{\mathrm{T}}$ and $\underline{c}_4 = [+, \pm, 0]^{\mathrm{T}}$ in this case. There are 4 possible cases for the entries $(2, 6)$ and $(2, 7)$. First we will examine the case where the entry $(2, 6)$ is -1 and $(2, 7)$ is $+1$.

From the order of the matrix W and the orthogonality of its three first rows we get the following system of four equations:

$$\begin{cases} u + v + x + y = n - 7, \\ u + v - x - y = 1, \\ u - v + x - y = -1, \\ u - v - x + y = -1, \end{cases}$$

which has the exact solution $(u, v, x, y) = \frac{1}{4}(n - 8, n - 4, n - 8, n - 8)$.

According to the definition of the $W(n, n - 2)$, the $(n - 3) \times (n - 3)$ matrix $D^{\mathrm{T}}D$ has the form $D^{\mathrm{T}}D = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}$, where

$$A = \begin{bmatrix} n-3 & -1 & 0 & 0 \\ -1 & n-3 & 0 & 0 \\ 0 & 0 & n-4 & 0 \\ 0 & 0 & 0 & n-4 \end{bmatrix}, \quad B = \begin{bmatrix} \overbrace{-1}^{u} & \overbrace{1}^{v} & \overbrace{-1}^{x} & \overbrace{1}^{y} \\ -1 & 1 & -1 & 1 \\ 0 & 0 & -2 & -2 \\ -2 & -2 & 0 & 0 \end{bmatrix},$$

$$C=\begin{bmatrix} C_{1u\times u} & -1_{u\times v} & -1_{u\times x} & 1_{u\times y}\\ -1_{v\times u} & C_{1v\times v} & 1_{v\times x} & -1_{v\times y}\\ -1_{x\times u} & 1_{x\times v} & C_{1x\times x} & -1_{x\times y}\\ 1_{y\times u} & -1_{y\times v} & -1_{y\times x} & C_{1y\times y}\end{bmatrix} \quad \text{and}$$

$$C_1=\begin{bmatrix} n-5 & -3 & \cdots & -3\\ -3 & n-5 & \cdots & -3\\ \vdots & \vdots & \ddots & \vdots\\ -3 & -3 & \cdots & n-5\end{bmatrix}.$$

According to (3), we have

$$\det D^{\mathrm{T}} D=\det A\cdot\det(C-B^{\mathrm{T}}A^{-1}B) \tag{10}$$

After the appropriate calculations we have

$$\det A=(n-4)^3(n-2) \quad \text{and} \quad C-B^{\mathrm{T}}A^{-1}B=\frac{n-2}{n-4}\begin{bmatrix} E_{u\times u} & F\\ F^{\mathrm{T}} & G\end{bmatrix}, \tag{11}$$

where E is a matrix of the form $E=(n-4)I-3J$,

$$F=\begin{bmatrix} -1_{u\times v} & -1_{u\times x} & 1_{u\times y}\end{bmatrix} \quad \text{and} \quad G=\begin{bmatrix} E_{v\times v} & 1_{v\times x} & -1_{v\times y}\\ 1_{x\times v} & E_{x\times x} & -1_{x\times y}\\ -1_{y\times v} & -1_{y\times x} & E_{y\times y}\end{bmatrix}.$$

So, according to (3),

$$\det(C-B^{\mathrm{T}}A^{-1}B)=\left(\frac{n-2}{n-4}\right)^{n-7}\det E_{u\times u}\cdot\det(G-F^{\mathrm{T}}E_{u\times u}^{-1}F). \tag{12}$$

From (1) we have

$$\det E_{u\times u}=\frac{n+8}{4}(n-4)^{\frac{n-12}{4}} \tag{13}$$

and from (2) $E_{u\times u}^{-1}=\frac{1}{(n-4)(n+8)}((n+8)I_u+12J_u)$.

Hence,

$$G-F^{\mathrm{T}}E_{u\times u}^{-1}F=\begin{bmatrix} K_{v\times v} & L_{v\times x} & -L_{v\times y}\\ L_{x\times v} & K_{x\times x} & -L_{x\times y}\\ -L_{y\times v} & -L_{y\times x} & K_{y\times y}\end{bmatrix}\equiv\begin{bmatrix} K_{v\times v} & N_2\\ N_2^{\mathrm{T}} & N_1\end{bmatrix},$$

where

$$K=(k_1-\lambda_1)I+\lambda_1 J, \quad k_1=\frac{(n-2)(n^2-48)}{(n-4)(n+8)}, \quad \lambda_1=-4\frac{(n-2)(n+4)}{(n-4)(n+8)}$$

$$L=\lambda_2 J, \quad \lambda_2=\frac{16(n-2)}{(n-4)(n+8)}.$$

The matrices I, J in the above assignments for E, K and L are of appropriate order.

So, according to (3), $\det(G-F^{\mathrm{T}}E_{u\times u}^{-1}F)=\det K_{v\times v}\cdot\det(N_1-N_2^{\mathrm{T}}K_{v\times v}^{-1}N_2)$.

We proceed in an absolute similar way like before in order to calculate $det K_{v\times v}$ and $\det(N_1-N_2^{\mathrm{T}}K_{v\times v}^{-1}N_2)$, by making use of (1)–(3).

We exploit again the appearing block structure and utilize equations (1)–(3) to facilitate our computations. After a lot of calculations we have

$$\det(G - F^{\mathrm{T}} E_{u\times u}^{-1} F) = \frac{64}{n+8}(n-4)^{\frac{3n}{4}-7}. \tag{14}$$

Finally, from (10)–(14), we have

$$\begin{aligned}\det D^{\mathrm{T}} D &= (n-4)^3(n-2)\left(\frac{n-2}{n-4}\right)^{n-7}\frac{n+8}{4}(n-4)^{\frac{n-12}{4}}\frac{64}{n+8}(n-4)^{\frac{3n}{4}-7}\\ &= 16(n-2)^{n-6}\end{aligned}$$

So, $\det D = 4(n-2)^{\frac{n}{2}-3}$.

Similarly we handle the other three possible cases for the entries (2, 6) and (2, 7) and we obtain the result $\det D = 4(n-2)^{\frac{n}{2}-3}$.

In a similar manner we deal with the other 60 cases and derive the values 0, $(n-2)^{\frac{n}{2}-3}$, $2(n-2)^{\frac{n}{2}-3}$, $3(n-2)^{\frac{n}{2}-3}$ and $4(n-2)^{\frac{n}{2}-3}$. For example, for the choice of $\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & + \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & + \\ + & + & 0 \\ + & + & + \end{bmatrix}$ as ULC's, the proof will start with W having as first three rows:

$$\left.\begin{array}{ccc} 0 & + & + \\ + & 0 & + \\ + & + & + \end{array}\right| \begin{array}{ccccllll} 0 & + & + & + & \overbrace{+\cdots+}^{u} & \overbrace{+\cdots+}^{v} & \overbrace{+\cdots+}^{x} & \overbrace{+\cdots+}^{y} \\ + & 0 & \pm & \pm & +\cdots+ & +\cdots+ & -\cdots- & -\cdots- \\ \pm & \pm & 0 & 0 & +\cdots+ & -\cdots- & +\cdots+ & -\cdots- \end{array}$$

and

$$\left.\begin{array}{ccc} 0 & 0 & + \\ + & + & 0 \\ + & + & + \end{array}\right| \begin{array}{cccllll} + & + & + & \overbrace{+\cdots+}^{u} & \overbrace{+\cdots+}^{v} & \overbrace{+\cdots+}^{x} & \overbrace{+\cdots+}^{y} \\ 0 & \pm & \pm & +\cdots+ & +\cdots+ & -\cdots- & -\cdots- \\ \pm & 0 & 0 & +\cdots+ & -\cdots- & +\cdots+ & -\cdots- \end{array}$$

respectively, and following an absolutely similar procedure the rest of the proof is carried out. These cases give the results $(n-2)^{\frac{n}{2}-3}$ and 0, respectively. $\square$

3. Existence of submatrices and specification of values of minors for $W(n, n-2)$

3.1. Counting techniques for specifying the existence of submatrices inside a $W(n, n-2)$

In several applications of the weighing matrices, e.g. Numerical Analysis, it is useful to specify the existence of certain submatrices having a required property, such as maximum determinant, inside a given $W(n, n-k)$. In this section we demonstrate with help of a counting technique the existence of specific submatrices embedded inside a $W(n, n-2)$. We would like to emphasize that results for other values of k fixed can be derived similarly and we treat here only the case $k = 2$ for an overview.

We can assume without loss of generality that the 2×2 matrix $[\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}]$ will always occur in the upper left corner of a $W(n, n-k)$ due to the orthogonality of the first two rows. Even if it does not exist initially in the upper left 2×2 corner, it can appear there with H-equivalence operations. This 2×2 matrix has maximum determinant over all possible 2×2 matrices with entries $(0, +, -)$. We are interested in specifying embedded matrices with maximum determinant, so that the CP property is attained. We extend this 2×2 matrix to all possible $(0, +, -)$ 3×3 and afterwards 4×4 matrices with maximum determinant values. The representatives (in the sense of

H-equivalence) of their extensions are named B_1, B_2 and A_1, A_2, A_3, respectively, and are used in the sequel. This idea with the extensions is presented for first time in [8] and is described there in detail for $W(n, n-1)$.

Lemma 2. *H-equivalence operations can be used to ensure that the following submatrices always occur in a* $W(n, n-2)$ *for large enough* n:

$$B_1 = \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix} \quad or \quad B_2 = \begin{bmatrix} + & + & + \\ + & - & 0 \\ + & + & - \end{bmatrix}.$$

Proof. The first three rows of a $W = W(n, n-2)$ can be written in the following form (cf. the comments at the beginning of the proof of Proposition 3)

$$W = \left[\begin{array}{cc|cccc|cccc} + & + & \overbrace{+}^{x_1} & \overbrace{+}^{x_2} & \overbrace{+}^{x_3} & \overbrace{+}^{x_4} & 0 & 0 & + & + \\ + & - & + & + & - & - & 0 & 0 & a & b \\ \hline + & z & + & - & + & - & + & + & 0 & 0 \end{array}\right],$$

where the parameters a, b, z can be ± 1. The previous form of W denotes that the column $[+, +, +]^{\mathrm{T}}$ occurs x_1 times, $[+, +, -]^{\mathrm{T}}$ occurs x_2 times etc.

From the order of W and the orthogonality of its first three rows we have

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = n - 6, \\ x_1 + x_2 - x_3 - x_4 = -a - b, \\ x_1 - x_2 + x_3 - x_4 = -1 - z, \\ x_1 - x_2 - x_3 + x_4 = -1 + z. \end{cases}$$

The solution of the system is

$$(x_1, x_2, x_3, x_4) = \left(\frac{n-a-b}{4} - 2, \frac{n-a-b}{4} - 1, \frac{n+a+b}{4} - \frac{z}{2} - \frac{3}{2}, \frac{n+a+b}{4} + \frac{z}{2} - \frac{3}{2}\right).$$

For $z = 1$ we see that the first two columns of B_1 exist and we need to prove that the third exists as well. It is sufficient to show that $x_4 \geqslant 1$.

We have $4x_4 = n + a + b - 4 \geqslant n - 6$. If we assume $n \geqslant 8$, we obtain $4x_4 \geqslant 2 \Leftrightarrow x_4 \geqslant \frac{1}{2}$, which means actually $x_4 \geqslant 1$, since x_4 denotes number of columns and must be integer.

For $z = -1$ there exist the first and the third column of B_1 and we must prove the existence of its second column, i.e. $x_3 \geqslant 1$. Absolutely similarly to the case $z = 1$, we prove for $z = -1$ that it holds $x_3 \geqslant 1$, when $n \geqslant 8$. Finally, we have that B_1 exists in any $W(n, n-2)$ for $n \geqslant 8$.

In order to prove the existence of B_2 we write the first three rows of a $W(n, n-2)$ in the following form

$$W = \left[\begin{array}{cc|ccccc|ccc} + & + & + & \overbrace{+}^{x_1} & \overbrace{+}^{x_2} & \overbrace{+}^{x_3} & \overbrace{+}^{x_4} & 0 & 0 & + \\ + & - & 0 & + & + & - & - & + & + & 0 \\ \hline + & z & d & + & - & + & - & 0 & 0 & c \end{array}\right],$$

where $c, d, z = \pm 1$. We set up the system as before and after examining all possible values for c, d, z, we obtain finally that B_2 exists in any $W(n, n-2)$ for $n \geqslant 8$. $\square$

Lemma 3. *H-equivalence operations can be used to ensure that the following submatrices always occur in a $W(n, n-2)$ for large enough n:*

$$A_1=\begin{bmatrix}+&+&+&+\\+&-&-&+\\+&+&-&-\\+&-&+&-\end{bmatrix}\quad or\quad A_2=\begin{bmatrix}+&+&+&+\\+&-&-&0\\+&+&-&-\\+&-&+&-\end{bmatrix}\quad or$$

$$A_3=\begin{bmatrix}+&+&+&+\\+&-&-&0\\+&+&-&-\\+&0&+&-\end{bmatrix}.$$

Proof. The first four rows of a $W = W(n, n-2)$ can be written in the following form (follows as an extension of the result of Corollary 1 and from the comments at the beginning of the proof of Proposition 3), since B_1 was proved to exist:

$$W=\left[\begin{array}{ccc|cccccccc|cccc} +&+&+&\overbrace{+}^{x_1}&\overbrace{+}^{x_2}&\overbrace{+}^{x_3}&\overbrace{+}^{x_4}&\overbrace{+}^{x_5}&\overbrace{+}^{x_6}&\overbrace{+}^{x_7}&\overbrace{+}^{x_8}&0&0&+&+\\ +&-&-&+&+&+&+&-&-&-&-&0&0&a&b\\ +&+&-&+&+&-&-&+&+&-&-&+&+&0&0\\ \cline{1-3} +&z&w&+&-&+&-&+&-&+&-&c&d&0&0 \end{array}\right],$$

where $a, b, c, d, z, w = \pm 1$. Similarly to the proof of Lemma 2, we obtain that A_1 always exists in a $W(n, n-2)$ for $n \geqslant 14$. Writing appropriately the first four rows and working similarly yields that A_2 and A_3 always exist in a $W(n, n-2)$ for $n \geqslant 10$. $\square$

Remark 4. For smaller n than the ones obtained in Lemmas 2 and 3, we can prove explicitly the existence of these matrices, as it is done in Lemma 4.

Remark 5. Counting techniques, like in Lemmas 2 and 3, can be used also for proving the non-existence of a matrix A for specific n. In such a case, the system with unknowns x_i will give as solution either zero for the x_i representing the columns of A or non-integers or negative values, which are not an acceptable solution, because x_i stand for numbers of columns and must be non-negative integers.

Lemma 4. *H-equivalence operations can be used to ensure that B_2 and A_1 (as defined previously) can always occur in the upper left corner of a $W(6, 4)$, and of a $W(10, 8)$ and $W(12, 10)$, respectively.*

Proof. Consider the following $W(6, 4)$

$$\begin{bmatrix}+&+&+&0&-&0\\+&-&0&+&0&+\\+&+&-&0&+&0\\0&0&+&+&+&-\\0&0&+&-&+&+\\+&-&0&-&0&-\end{bmatrix}.$$

We can see B_2 in the upper left 3×3 corner of it.

Now consider the following $W(10, 8)$

$$\begin{bmatrix}
+ & + & + & + & + & + & 0 & + & 0 & + \\
+ & - & - & + & - & + & 0 & + & 0 & - \\
+ & + & - & - & 0 & + & - & - & - & 0 \\
+ & - & + & - & 0 & + & + & - & + & 0 \\
- & 0 & 0 & - & + & + & - & + & + & - \\
- & 0 & 0 & + & - & + & - & - & + & + \\
- & - & - & 0 & + & + & + & 0 & - & + \\
+ & - & - & 0 & + & - & - & 0 & + & + \\
0 & - & + & - & - & 0 & - & + & - & + \\
0 & + & - & - & - & 0 & + & + & + & +
\end{bmatrix}.$$

We can see A_1 in the upper left 4×4 corner of it.
A_1 can be seen also in the upper left corner of the following $W(12, 10)$

$$\begin{bmatrix}
+ & + & + & + & 0 & 0 & + & + & + & + & + & + \\
+ & - & - & + & + & + & 0 & 0 & + & + & - & - \\
+ & + & - & - & 0 & 0 & + & - & - & + & - & + \\
+ & - & + & - & - & + & + & - & + & - & 0 & 0 \\
+ & + & - & + & - & - & - & - & + & - & 0 & 0 \\
+ & 0 & - & - & + & + & - & + & 0 & - & + & + \\
+ & 0 & + & - & + & - & - & - & 0 & + & + & - \\
- & - & - & 0 & + & - & + & - & + & 0 & + & + \\
0 & - & 0 & + & - & + & - & - & - & + & + & + \\
- & + & - & - & - & + & 0 & 0 & + & + & + & - \\
0 & + & 0 & + & + & + & + & - & - & - & + & - \\
+ & - & - & 0 & - & - & + & + & - & 0 & + & -
\end{bmatrix}. \qquad \square$$

3.2. *Specification of values of minors for $W(n, n-2)$*

Generally, it is interesting to specify all possible values of determinants for $(0, \pm 1)$ matrices. For the case of ± 1 matrices the following Proposition describes the possible range of values.

Proposition 4. [3] *Let B be an $n \times n$ matrix with elements ± 1. Then*

(i) $\det B$ *is an integer and 2^{n-1} divides* $\det B$;
(ii) *when $n \leqslant 6$, the only possible values for* $\det B$ *are in Table 1, and they do all occur.*

We extended the above results for the 7×7 case in the following Proposition 5.

Proposition 5. *The possible values for the determinant of a 7×7 matrix with elements ± 1 are the following and they do all occur*: 0, 64, 128, 192, 256, 320, 384, 448, 512, 576.

Table 1
Possible determinant values for $n \times n$ ± 1 matrices

n	1	2	3	4	5	6
$\det B$	1	0, 2	0, 4	0, 8, 16	0, 16, 32, 48	0, 32, 64, 96, 128, 160

Proof. We constructed an algorithm on the computer, which performs an exhaustive search over all possible values of some entries of an $(1, -1)$ 7×7 matrix A, while keeping the upper left 4×4 corner of it fixed and having the elements of its first row and column equal to 1. We can assume without loss of generality the elements of the first row and column to be 1, because we can always have this form with column and/or row multiplications by -1, which do not affect the absolute value of the determinant.

Keeping the upper left 4×4 corner of the matrix fixed we achieved to obtain the desired results with a significant smaller amount of searches, than the one needed, if we would perform an exhaustive search over all entries of the 6×6 lower right matrix. Precisely, such an exhaustive search would need $2^{36} \simeq 10^{10}$ comparisons, while our idea requires $2^{27} \simeq 10^8$ searches, which is a much smaller quantity. We have been led intuitively to select as upper left 4×4 corner the matrix

$$\begin{bmatrix} + & + & + & + \\ + & - & - & + \\ + & + & - & - \\ + & - & + & - \end{bmatrix},$$

which is familiar due to its existence properties concerning Hadamard [9] and weighing matrices [8].

So, the algorithm operates according to the following scheme:

$$A = \begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & * & * & * \\ + & + & - & - & * & * & * \\ + & - & + & - & * & * & * \\ + & * & * & * & * & * & * \\ + & * & * & * & * & * & * \\ + & * & * & * & * & * & * \end{bmatrix},$$

where the elements $*$ take the values ± 1.

The algorithm gave as a result matrices attaining all the values of determinants given in the enunciation of the Proposition. In the Appendix we provide exemplarily one matrix for each determinant value.

Since we have not performed an exhaustive search over all possible entries of an $(1, -1)$ 7×7 matrix, but only to a subset of these values, in order to complete the proof, we need to show also that the obtained values are the only possible values for the determinant of an $(1, -1)$ 7×7 matrix, i.e. there do not exist any other values.

From Proposition 5, (i) we have that $2^6 = 64$ divides $\det A$, so we have that the possible values for $\det A$, which are $\leqslant 576$, are only the ones given in the enunciation, since they are all the multiplies of 64 less or equal than 576. It is sufficient now to show that there do not exist any other values for $\det A$ greater than 576. Williamson in [14] showed that the maximum value of the determinant of an $(1, -1)$ matrix of order 7 is $2^6 \cdot 9 = 576$. This result completes our proof. $\square$

In the above demonstrated proof it was very important to find an idea that could give all the desired determinant values and to avoid the exhaustive search over all possible entries of the lower right 6×6 matrix. The significance of such an idea is confirmed by Table 2, which gives the estimated time needed for all the searches (according to a specific computer capabilities), when the upper left $j \times j$ corner is fixed, and also the first row and column as mentioned before.

Table 2
Time needed with $j \times j$ upper left corner fixed, for the 7×7 case

j	1	2	3	4	5	6
Time	572 h	353 h	44 h	1 h 20′	31′	0.2″

Table 3
Time needed with $j \times j$ upper left corner fixed, for the 8×8 case

j	1	2	3	4	5	6	7
Time	4×10^6 h	2×10^6 h	3×10^5 h	9×10^3 h	71 h 30′	500″	1″

We note that the time needed for $j = 5, 6$ is very small, but we could not obtain all the determinant values, which seems to be logical with an intuitive sense. Although the time needed for $j = 1, 2, 3$ is a lot, but not totally impossible, we always need an optimal time interval for carrying out the computations on the computer and we believe that the time needed for the case $j = 4$ is realistic.

Table 3 shows the corresponding times needed for the 8×8 case, reveals the computational difficulties on how to obtain such results.

In this case, the selections of $j = 6, 7$ do not provide all the results as expected. The choice of $j = 5$, which is still realizable, does not give all the values, which are expected to appear according to Conjecture 1. For $j \leqslant 4$, the time needed is absolutely inconvenient, as e.g. for $j = 4$ we would need about 375 days. So, we are led to believe that the result should be found with $j = 5$, but, of course, the appropriate choice (or choices) for the upper left 5×5 corner are needed.

Propositions 4 and 5 led us to posing the following conjecture.

Conjecture 1. *The determinant of an $n \times n$ matrix with elements ± 1 is 0 or p, for $p = 2^{n-1}, 2 \cdot 2^{n-1}, 3 \cdot 2^{n-1}, \dots, s \cdot 2^{n-1}$, where $s \cdot 2^{n-1} = \max\{\det(A) | A \in \mathbb{R}^{n \times n},\ \text{with elements}\ \pm 1\}$ and the value 0 is excluded from the case $n = 1$.*

For the case of weighing matrices, the existence of the submatrices in the previous Section 3.1 can help us to specify values of principal minors as follows. From Lemmas 2, 3 and 4 we conclude Corollary 2.

Corollary 2. *$W(3) = 4$ for a CP $W(n, n-2)$ with $n \geqslant 6$ and $W(4) = 16$ or 12 or 10 for a CP $W(n, n-2)$ with $n \geqslant 10$.*

Next, we tried to extend the 4×4 matrices A_1, A_2 and A_3 to all possible 5×5 matrices with elements $(0, +, -)$ with the restriction that every row and column contains at most two zeros, according to the property of the $W(n, n-2)$. For this purpose were created appropriate algorithms on the computer. It is interesting to specify all possible 5×5 matrices that contain the matrices A_1 or A_2 or A_3 and also have the maximum possible values of the determinant. These maximum values can be easily found by computer by assigning to all 25 entries of a 5×5 matrix all possible values $(0, +, -)$, with the restriction that every row and column contains at most 2 zeros, as this 5×5 matrix is supposed to be a submatrix of a $W(n, n-2)$. We found the results presented in Table 4, which shows the number of matrices that occurred as extensions of A_1, A_2 and A_3, respectively, with the corresponding determinant values.

Table 4
Extensions of matrices A_1, A_2 and A_3

Extensions of matrix A_1	det	20	25	27	28	30	32	36	40	48
	matrices	81	0	0	63	0	211	27	24	4
Extensions of matrix A_2	det	20	25	27	28	30	32	36	40	48
	matrices	97	0	0	40	10	23	10	4	0
Extensions of matrix A_3	det	20	25	27	28	30	32	36	40	48
	matrices	178	5	1	18	6	6	2	0	0

An algorithm extending $(0, +, -)$ *matrices to* $W(n, n-k)$(Algorithm Extend). We need to find a way to demonstrate that specific $j \times j$ matrices with known determinant can always exist embedded inside a $W(n, n-k)$. The idea is to create an algorithm, which extends a $j \times j$ $(0, +, -)$ matrix to a $W(n, n-k)$, if possible. If such an extension is successful, this means that the initial $j \times j$ matrix exists embedded inside a $W(n, n-k)$ for some specific n, and we can say that the $j \times j$ minor of this $W(n, n-k)$ is equal to the determinant of this $j \times j$ matrix. Otherwise, the $j \times j$ matrix cannot exist inside a $W(n, n-k)$. In [8] was given the Algorithm Extend for realizing this idea for a $W(n, n-1)$. For the purposes of this paper we had to change some features of the algorithm, so that it can be reapplied to work on a $W(n, n-k)$.

Using the above tools we can prove the following propositions:

Proposition 6. *$W(4) = 16$ or 10 for a CP $W(8, 6)$.*

Proof. We must show that only the 4×4 matrices with determinant 16 or 10, which are actually A_1 and A_3 respectively, can be extended to a $W(8, 6)$. By using Algorithm Extend with parameters $j = 4$, $k = 2$ and $n = 8$ and by testing the matrices A_1, A_2 and A_3, we found that only A_1 and A_3 can be extended to a $W(8, 6)$. Hence, since we have that A_1 and A_3 always exist in a $W(8, 6)$, we can conclude that $W(4) = 16$ or 10 for a CP $W(8, 6)$. □

Proposition 7. *$W(5) = 32$ or 28 or 20 and $W(6) = 128$ or 112 or 96 or 80 for a CP $W(10, 8)$.*

Proof. We must show that from all the 5×5 matrices in Table 4 only the ones with determinant 32 or 28 or 20 can be extended to a $W(10, 8)$. By using Algorithm Extend with parameters $j = 5$, $k = 2$ and $n = 10$ and by testing all the appearing matrices, we found that only five matrices with determinants 32, 20, 32, 28 and 32 can be extended to a $W(10, 8)$. Hence, since we have that these matrices always exist in a $W(10, 8)$, we can conclude that $W(5) = 32$ or 28 or 20 for a $W(10, 8)$. Similarly, we can conclude the result about $W(6)$. □

Similarly, we can derive the following result.

Proposition 8. *$W(5) = 25$ or 28 or 30 or 32 or 36 or 48, $W(6) = 100$ or 104 or 112 or 120 or 125 or 128 or 130 or 144 or 160 and $W(7) = 250$ or 300 or 320 or 360 or 480 for a CP $W(12, 10)$.*

Remark 6. Although the Counting Techniques presented in Section 3.1 contained easy calculations done by hand, they will feature more demanding computations in the cases of specifying the existence of a $j \times j$ matrix, $j \geqslant 5$, inside a weighing matrix of order $n \geqslant 14$. Therefore it is sensible to create an algorithmic version of them that will make use of the notion of symbolic manipulation. Such an implementation would overcome the obvious computational difficulties arising from Algorithm Extend and might be very helpful for approaching related problems involving a high complexity, such as the specification of the pivot pattern of the Hadamard matrix of order 16, which still remains an open problem.

4. An algorithm for finding minors of $W(n, n-2)$

4.1. The determinant simplification theorem

Notation. We write $J_{b_1,b_2,\ldots,b_z}$ for the all ones matrix with diagonal blocks of sizes $b_1 \times b_1, b_2 \times b_2 \cdots b_z \times b_z$, and $a_{ij}J_{b_1,b_2,\ldots,b_z}$ for the matrix, for which the elements of the block with corners $(i + b_1 + b_2 + \cdots + b_{j-1}, i + b_1 + b_2 + \cdots + b_{i-1})$, $(i + b_1 + b_2 + \cdots + b_{j-1}, b_1 + b_2 + \cdots + b_i)$, $(b_1 + b_2 + \cdots + b_j, i + b_1 + b_2 + \cdots + b_{i-1})$, $(b_1 + b_2 + \cdots + b_j, b_1 + b_2 + \cdots + b_i)$ are a_{ij} an integer.

We write $(k_i - a_{ii})I_{b_1,b_2,\ldots,b_z}$ for the direct sum $(k_1 - a_{11})I_{b_1} + (k_2 - a_{22})I_{b_2} + \cdots + (k_z - a_{zz})I_{b_z}$.

Example 3. According to this notation we have that the matrix

$$A = \begin{bmatrix} k & a & b & b & b \\ a & k & b & b & b \\ b & b & k & a & a \\ b & b & a & k & a \\ b & b & a & a & k \end{bmatrix},$$

can be written as $A = (k - a_{ii})I_{2,3} + a_{ij}J_{2,3}$, where$(a_{ij}) = [\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}]$.

Theorem 1 (Determinant Simplification Theorem). *Let* $A = (k_i - a_{ii})I_{b_1,b_2,\ldots,b_z} + a_{ij}J_{b_1,b_2,\ldots,b_z}$, $i, j = 1, \ldots, z$, *then*

$$\det A = \prod_{i=1}^{z}(k_i - a_{ii})^{b_i - 1} \det D,$$

where

$$D = \begin{bmatrix} k_1 + (b_1 - 1)a_{11} & b_2a_{12} & b_3a_{13} & \cdots & b_za_{1z} \\ b_1a_{21} & k_2 + (b_2 - 1)a_{22} & b_3a_{23} & \cdots & b_za_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ b_1a_{z1} & b_2a_{z2} & b_3a_{z2} & \cdots & k_z + (b_z - 1)a_{zz} \end{bmatrix}.$$

4.2. Algorithm minors

We will present an algorithm for calculating $(n - j) \times (n - j)$ minors of a $W(n, n-2)$. Any matrix $W = W(n, n-2)$ can be written according to the following two cases (it follows from an extension of the result of Corollary 1), as it is also verified in Example 4.

Example 4.

$$W(6,4)=\begin{bmatrix}0&0&+&+&+&+\\0&0&+&+&-&-\\+&+&0&0&+&-\\+&+&0&0&-&+\\+&-&+&-&0&0\\+&-&-&+&0&0\end{bmatrix}.$$

Considering the above $W(6,4)$ we see that for j even there are $j/2$ 2×2 blocks with zeros in the upper left $j\times j$ corner M, while for j odd there are $(j-1)/2$ and there is also a zero entry in its lower right corner.

First Case, $j\equiv 0 \bmod 2$ (j even). $W=\begin{bmatrix}M&U_j\\U_j^{\mathrm{T}}&C\end{bmatrix}$.

M, C are $j\times j$ and $(n-j)\times(n-j)$ matrices respectively. M has $j/2$ 2×2 blocks of zeros on the diagonal. The elements in the $(n-j)\times(n-j)$ matrix CC^{T}, where C is obtained by removing the first j rows and columns of the weighing matrix W, can be permuted to appear in the form

$$CC^{\mathrm{T}}=(n-2-j-a_{ii})I_{u_1,u_2,\ldots,u_{2^{j-1}}}+a_{ik}J_{u_1,u_2,\ldots,u_{2^{j-1}}},$$

where $(a_{ik})=(-\underline{u}_i\cdot\underline{u}_k)$, $a_{ii}=(-\underline{u}_i\cdot\underline{u}_i)=-j$, with $\cdot$ the inner product ($\underline{u}_i$ denotes the ith column of U_j, as it is described in the Notation at the end of Section 1). By the Determinant Simplification Theorem

$$\det CC^{\mathrm{T}}=(n-2)^{n-2^{j-1}-j}\det D,$$

where D, of order 2^{j-1}, is given by

$$D=\begin{bmatrix}n-2-ju_1&u_2a_{12}&u_3a_{13}&\cdots&u_za_{1z}\\u_1a_{21}&n-2-ju_2&u_3a_{23}&\cdots&u_za_{2z}\\\vdots&\vdots&\vdots&&\vdots\\u_1a_{z1}&u_2a_{z2}&u_3a_{z2}&\cdots&n-2-ju_z\end{bmatrix},$$

where $z=2^{j-1}$.

The $(n-j)\times(n-j)$ minor of W is the determinant of C, for which we have

$$\det C=((n-2)^{n-2^{j-1}-j}\det D)^{1/2}. \tag{15}$$

Second case, $j\equiv 1 \bmod 2$ (j odd). $W=\begin{bmatrix}M&\underline{v}&U_j\\\underline{v}^{\mathrm{T}}&&\\U_j^{\mathrm{T}}&&C\end{bmatrix}$.

M, C are $j\times j$ and $(n-j)\times(n-j)$ matrices respectively. M has $(j-1)/2$ 2×2 blocks of zeros on the diagonal and one zero element in the lower right entry. The vector $\underline{v}$ of order $j\times 1$ is of the form $[v^{(j-1)}\ 0]^{\mathrm{T}}$, where $v^{(j-1)}$ is a possible column of U_{j-1}. The elements in the $(n-j)\times(n-j)$ matrix CC^{T} can be permuted to appear in the form

$$CC^{\mathrm{T}}=\begin{bmatrix}n-1-j&\underline{y}\\\underline{y}^{\mathrm{T}}&E\end{bmatrix},$$

where $E=(n-2-j-a_{ii})I_{u_1,u_2,\ldots,u_{2^{j-1}}}+a_{ik}J_{u_1,u_2,\ldots,u_{2^{j-1}}}$, $(a_{ik})=(-\underline{u}_i\cdot\underline{u}_k)$, with $\cdot$ the inner product, and $\underline{y}$ is a vector of order $1\times(n-j-1)$, whose elements are obtained from the inner products of $\underline{v}$ with $\underline{u}_i$.

Precisely, we have

$$\begin{aligned}\underline{y} &= [\underbrace{-(\underline{v}\cdot\underline{u}_1)\ldots-(\underline{v}\cdot\underline{u}_1)}_{u_1\text{ times}}\underbrace{-(\underline{v}\cdot\underline{u}_2)\ldots-(\underline{v}\cdot\underline{u}_2)}_{u_2\text{ times}}\ldots\underbrace{-(\underline{v}\cdot\underline{u}_{2^{j-1}})\ldots-(\underline{v}\cdot\underline{u}_{2^{j-1}})}_{u_{2^{j-1}}\text{ times}}]\\ &= [\underbrace{b_1\ldots b_1}_{u_1}\underbrace{b_2\ldots b_2}_{u_2}\ldots\underbrace{b_z\ldots b_z}_{u_z}],\end{aligned}$$

where $b_i = (-\underline{v}\cdot\underline{u}_i)$ and $z = 2^{j-1}$.
We want to calculate $\det CC^{\mathrm{T}}$ with help of formula (3). So, we have

$$\det CC^{\mathrm{T}} = (n-1-j)\cdot\det\left(E - \frac{1}{n-1-j}\underline{y}^{\mathrm{T}}\underline{y}\right).$$

We have $\underline{y}^{\mathrm{T}}\underline{y} = \gamma_{ik}J_{u_1,u_2,\ldots,u_z}$, where $\gamma_{ik} = b_ib_k$.

$$\begin{aligned}X &\equiv E - \frac{1}{n-1-j}\underline{y}^{\mathrm{T}}\underline{y}\\ &= (n-2-j-a_{ii})I_{u_1,u_2,\ldots,u_{2^{j-1}}} + a_{ik}J_{u_1,u_2,\ldots,u_{2^{j-1}}} - \frac{1}{n-1-j}\gamma_{ik}J_{u_1,u_2,\ldots,u_z}\\ &= (n-2-j-a_{ii})I_{u_1,u_2,\ldots,u_{2^{j-1}}} + \left(a_{ik} - \frac{1}{n-1-j}\gamma_{ik}\right)J_{u_1,u_2,\ldots,u_{2^{j-1}}}.\end{aligned}$$

We set $\delta_{ik} = \frac{1}{n-1-j}\gamma_{ik}$. For the sake of simplicity we omit the subscripts $u_1,\ldots,u_{2^{j-1}}$. Hence

$$\begin{aligned}X &= (n-2-j-a_{ii})I + (a_{ik}-\delta_{ik})J\\ &= [n-2-j-\delta_{ii}-(a_{ii}-\delta_{ii})]I + (a_{ik}-\delta_{ik})J\\ &= (\lambda_i-\varepsilon_{ii})I + \varepsilon_{ik}J,\end{aligned}$$

where $\lambda_i = n-2-j-\delta_{ii}$ and $\varepsilon_{ik} = a_{ik}-\delta_{ik}$.
By the Determinant Simplification Theorem

$$\det X = \prod_{i=1}^{z}(\lambda_i-\varepsilon_{ii})^{u_i-1}\det D, \tag{16}$$

where

$$D = \begin{bmatrix}\lambda_1+(u_1-1)\varepsilon_{11} & u_2\varepsilon_{12} & u_3\varepsilon_{13} & \cdots & u_z\varepsilon_{1z}\\ u_1\varepsilon_{21} & \lambda_2+(u_2-1)\varepsilon_{22} & u_3\varepsilon_{23} & \cdots & u_z\varepsilon_{2z}\\ \vdots & \vdots & \vdots & & \vdots\\ u_1\varepsilon_{z1} & u_2\varepsilon_{z2} & u_3\varepsilon_{z2} & \cdots & \lambda_z+(u_z-1)\varepsilon_{zz}\end{bmatrix}.$$

Finally

$$\det C = ((n-1-j)\det X)^{1/2} \tag{17}$$

Remark 7. For $k = 1$ the algorithm has obviously one case. M, C are $j\times j$ and $(n-j)\times(n-j)$ matrices respectively and M has diagonal entries all zero. For CC^{T} we have always

$$CC^{\mathrm{T}} = (n-1-j-a_{ii})I_{u_1,u_2,\ldots,u_{2^{j-1}}} + a_{ik}J_{u_1,u_2,\ldots,u_{2^{j-1}}},$$

so

$$\det CC^{\mathrm{T}} = (n-1)^{n-2^{j-1}-j}\det D,$$

where

$$D=\begin{bmatrix} n-1-ju_1 & u_2a_{12} & u_3a_{13} & \cdots & u_za_{1z} \\ u_1a_{21} & n-1-ju_2 & u_3a_{23} & \cdots & u_za_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ u_1a_{z1} & u_2a_{z2} & u_3a_{z2} & \cdots & n-1-ju_z \end{bmatrix}.$$

Finally

$$\det C=((n-1)^{n-2^{j-1}-j}\det D)^{1/2}.$$

This algorithm can be also easily applied for the special case of a Hadamard matrix. In this case

$$\det C=(n^{n-2^{j-1}-j}\det D)^{1/2},$$

where

$$D=\begin{bmatrix} n-ju_1 & u_2a_{12} & u_3a_{13} & \cdots & u_za_{1z} \\ u_1a_{21} & n-ju_2 & u_3a_{23} & \cdots & u_za_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ u_1a_{z1} & u_2a_{z2} & u_3a_{z2} & \cdots & n-ju_z \end{bmatrix}.$$

For the appropriate implementation of the algorithm the following notion is required. The most practical way to manage the variables, which represent the unknown number of columns of U_j, is to denote with $u_l^{(s)}$, $l=1,\ldots,2^{k-1}$, $k=3,\ldots,j$, $s=1,\ldots,j-2$, the number of columns starting with the same vectors of order $s+2$. For example, for $j=5$, the matrix U_5 will be of the form:

$$\begin{array}{cccccccccccccccc}
\multicolumn{4}{c}{\overbrace{\hphantom{u_1^{(3)}\,u_2^{(3)}\,u_3^{(3)}\,u_4^{(3)}}}^{u_1^{(1)}}} & \multicolumn{4}{c}{\overbrace{\hphantom{u_5^{(3)}\,u_6^{(3)}\,u_7^{(3)}\,u_8^{(3)}}}^{u_2^{(1)}}} & \multicolumn{4}{c}{\overbrace{\hphantom{u_9^{(3)}\,u_{10}^{(3)}\,u_{11}^{(3)}\,u_{12}^{(3)}}}^{u_3^{(1)}}} & \multicolumn{4}{c}{\overbrace{\hphantom{u_{13}^{(3)}\,u_{14}^{(3)}\,u_{15}^{(3)}\,u_{16}^{(3)}}}^{u_4^{(1)}}} \\
\multicolumn{2}{c}{\overbrace{\hphantom{u_1^{(3)}\,u_2^{(3)}}}^{u_1^{(2)}}} & \multicolumn{2}{c}{\overbrace{\hphantom{u_3^{(3)}\,u_4^{(3)}}}^{u_2^{(2)}}} & \multicolumn{2}{c}{\overbrace{\hphantom{u_5^{(3)}\,u_6^{(3)}}}^{u_3^{(2)}}} & \multicolumn{2}{c}{\overbrace{\hphantom{u_7^{(3)}\,u_8^{(3)}}}^{u_4^{(2)}}} & \multicolumn{2}{c}{\overbrace{\hphantom{u_9^{(3)}\,u_{10}^{(3)}}}^{u_5^{(2)}}} & \multicolumn{2}{c}{\overbrace{\hphantom{u_{11}^{(3)}\,u_{12}^{(3)}}}^{u_6^{(2)}}} & \multicolumn{2}{c}{\overbrace{\hphantom{u_{13}^{(3)}\,u_{14}^{(3)}}}^{u_7^{(2)}}} & \multicolumn{2}{c}{\overbrace{\hphantom{u_{15}^{(3)}\,u_{16}^{(3)}}}^{u_8^{(2)}}} \\
u_1^{(3)} & u_2^{(3)} & u_3^{(3)} & u_4^{(3)} & u_5^{(3)} & u_6^{(3)} & u_7^{(3)} & u_8^{(3)} & u_9^{(3)} & u_{10}^{(3)} & u_{11}^{(3)} & u_{12}^{(3)} & u_{13}^{(3)} & u_{14}^{(3)} & u_{15}^{(3)} & u_{16}^{(3)} \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & + & + & + & - & - & - & - & + & + & + & - & - & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & + & - & + & - & - \\
+ & - & + & - & + & - & + & - & + & - & + & + & + & - & - & -
\end{array}$$

We see easily that the following relation connects the above numbers of columns:

$$u_{2l-1}^{(s+1)}+u_{2l}^{(s+1)}=u_l^{(s)},\quad l=1,2,\ldots,2^{j-1},\ s\geqslant 1. \tag{18}$$

The following algorithm calculates the value of the determinant of the $(n-j)\times(n-j)$ lower right submatrix of a $W(n,n-k)$.

Algorithm Minors

Step 1: **Read** all $j\times j$ matrices M, which can exist in the upper left corner of a $W(n,n-k)$

Step 2: **For every** matrix M

Create the $j\times n$ matrix $N=[M\ U_j]$, if j even, or $N=[M\ v\ U_j]$, if j odd

Step 3: $s:=0$

For $k=3,4,\ldots,j$

Step 4: Consider the first k rows of N

$s := s + 1$

Set $u_l^{(s)}$ the number of columns starting with the vectors $\underline{u}_l, l = 1, \ldots, 2^{k-1}$

Form the system resulting from orthogonality of rows and counting of columns, with unknowns $u_l^{(s)}$

Solve the system taking into account (18)

End {for $k = 3, \ldots, j$}

Step 5: **For every** acceptable solution $(u_1^{(j-2)}, \ldots, u_{2^{j-1}}^{(j-2)})$ of the system calculate the values of the $(n-j) \times (n-j)$ minors, using (15), or (16) and (17).

End {for every matrix M}

End {of Algorithm}

4.3. Applications of the algorithm minors

1. Application of the Algorithm Minors

We want to calculate the $n-4$ minor of a $W(n, n-2)$. After finding all possible $2^2 = 4$ matrices M, we create $N = [M\ U_4]$, where M is of the form

$$\begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & a \\ + & + & 0 & 0 \\ + & b & 0 & 0 \end{bmatrix},$$

with $a, b = \pm 1$.

For $k = 3$ the system is

$$\begin{cases} u_1^{(1)} + u_2^{(1)} + u_3^{(1)} + u_4^{(1)} = n - 4, \\ u_1^{(1)} + u_2^{(1)} - u_3^{(1)} - u_4^{(1)} = -1 - a, \\ u_1^{(1)} - u_2^{(1)} + u_3^{(1)} - u_4^{(1)} = 0, \\ u_1^{(1)} - u_2^{(1)} - u_3^{(1)} + u_4^{(1)} = 0, \end{cases}$$

with solution $(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_4^{(1)}) = (\frac{1}{4}(n-5-a), \frac{1}{4}(n-5-a), \frac{1}{4}(n-3+a), \frac{1}{4}(n-3+a))$.

For $k = 4$ the system is

$$\begin{cases} u_1^{(2)} + u_2^{(2)} + u_3^{(2)} + u_4^{(2)} + u_5^{(2)} + u_6^{(2)} + u_7^{(2)} + u_8^{(2)} = n - 4, \\ u_1^{(2)} + u_2^{(2)} + u_3^{(2)} + u_4^{(2)} - u_5^{(2)} - u_6^{(2)} - u_7^{(2)} - u_8^{(2)} = -1 - a, \\ u_1^{(2)} - u_2^{(2)} - u_3^{(2)} + u_4^{(2)} + u_5^{(2)} - u_6^{(2)} - u_7^{(2)} + u_8^{(2)} = 0, \\ u_1^{(2)} - u_2^{(2)} + u_3^{(2)} - u_4^{(2)} + u_5^{(2)} - u_6^{(2)} + u_7^{(2)} - u_8^{(2)} = 0, \\ u_1^{(2)} + u_2^{(2)} - u_3^{(2)} - u_4^{(2)} - u_5^{(2)} - u_6^{(2)} + u_7^{(2)} + u_8^{(2)} = 0, \\ u_1^{(2)} - u_2^{(2)} + u_3^{(2)} - u_4^{(2)} - u_5^{(2)} + u_6^{(2)} - u_7^{(2)} + u_8^{(2)} = 0, \\ u_1^{(2)} - u_2^{(2)} - u_3^{(2)} + u_4^{(2)} + u_5^{(2)} - u_6^{(2)} - u_7^{(2)} + u_8^{(2)} = -1 - b, \end{cases}$$

with solution $(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, u_4^{(2)}, u_5^{(2)}, u_6^{(2)}, u_7^{(2)}, u_8^{(2)}) = (\frac{1}{4}(n-5-b) - u_8^{(2)}, u_8^{(2)}, u_8^{(2)}, \frac{1}{4}(n-5-b) - u_8^{(2)}, u_8^{(2)}, \frac{1}{4}(n-3+a) - u_8^{(2)}, \frac{1}{4}(n-3+a) - u_8^{(2)}, u_8^{(2)})$.

Since $u_7^{(2)} + u_8^{(2)} = u_4^{(1)}$ and thus $u_8^{(2)} = u_4^{(1)} - u_7^{(2)} \leqslant u_4^{(1)}$ ($u_7^{(2)}$ is always a non-negative number), the range of values for $u_8^{(2)}$ is from 0 to $u_4^{(1)}$. We now compute for all the possible values of $u_8^{(2)}$ the acceptable solutions for the remaining $u_i^{(2)}$ and calculate the requested minor from (15).

For example, for $n = 12$, we have $0 \leqslant u_8^{(2)} \leqslant \frac{1}{4}(9 + a)$. For all possible values of a the upper bound is 2 ($u_8^{(2)}$ must be an integer), so for $u_8^{(2)} = 0, 1, 2$ we find the possible values for the rest of $u_i^{(2)}$ and finally apply formula (15). The resulting value for the 8×8 minor of the $W(12, 10)$, if we have 2×2 blocks with zeros on the diagonal of M, is always 400.

2. Application of the Algorithm Minors (modified)

With this application we show how Algorithm Minors can be easily modified for the case when we do not have the upper left $j \times j$ submatrix M in the form described in subsection 4.2. We are interested in specifying the minor $W(9)$ of a $W(12, 10)$, when the matrix contains in its upper left 3×3 corner B_1 (this is possible according to Lemma 2). The matrix $W = W(12, 10)$ can be written in the following form:

$$W = \left[\begin{array}{ccc|cccccccc}
 & & & & & & & \overbrace{}^{u_1} & \overbrace{}^{u_2} & \overbrace{}^{u_3} & \overbrace{}^{u_4} \\
+ & + & + & 0 & 0 & + & + & + & + & + & + \\
+ & - & - & 0 & 0 & a & b & + & + & - & - \\
+ & + & - & + & + & 0 & 0 & + & - & + & - \\
\hline
0 & 0 & + & & & & & & & & \\
0 & 0 & + & & & & & & & & \\
+ & a & 0 & & & & & & & & \\
+ & b & 0 & & & & & C & & & \\
+ & + & + & & & & & & & & \\
+ & + & - & & & & & & & & \\
+ & - & + & & & & & & & & \\
+ & - & - & & & & & & & &
\end{array}\right],$$

where $a, b = \pm 1$ and the first three columns contain also u_1 times the vector $[+, +, +]$, u_2 times the vector $[+, +, -]$ etc.

For $a = 1, b = -1$ we can find the exact number of columns u_i as solution of the system:

$$\begin{cases} u_1 + u_2 + u_3 + u_4 = 5, \\ u_1 + u_2 - u_3 - u_4 = 1, \\ u_1 - u_2 + u_3 - u_4 = -1, \\ u_1 - u_2 - u_3 + u_4 = -1. \end{cases}$$

The solution is $u_1 = 1, u_2 = 2, u_3 = 1$ and $u_4 = 1$. For $a = -1, b = 1$ we find the same solution, while for the other two cases of a, b there is not an acceptable solution.

From the properties of the $W(12, 10)$ we see easily that:

$$CC^{\mathrm{T}} = \left[\begin{array}{cccc|ccccc}
9 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 1 \\
-1 & 9 & 0 & 0 & -1 & 1 & 1 & -1 & 1 \\
0 & 0 & 8 & 0 & -2 & -2 & -2 & 2 & 2 \\
0 & 0 & 0 & 8 & -2 & -2 & -2 & 2 & 2 \\
\hline
-1 & -1 & -2 & -2 & 7 & -1 & -1 & -1 & 1 \\
1 & 1 & -2 & -2 & -1 & 7 & -3 & 1 & -1 \\
1 & 1 & -2 & -2 & -1 & -3 & 7 & 1 & -1 \\
-1 & -1 & 2 & 2 & -1 & 1 & 1 & 7 & -1 \\
1 & 1 & 2 & 2 & 1 & -1 & -1 & -1 & 7
\end{array}\right].$$

With help of formula (3) we can calculate the determinant of CC^{T}, which is 16,000,000, hence $\det C \equiv W(9) = 4000$.

5. Application to the growth problem

5.1. Description of the problem

Traditionally, backward error analysis for GE on a matrix $A = (a_{ij}^{(0)})$ is expressed in terms of the *growth factor*

$$g(n, A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}^{(0)}|},$$

which involves all the elements $a_{ij}^{(k)}$, $k = 0, 1, 2, \ldots, n-1$ that occur during the elimination. For a CP matrix A we have

$$g(n, A) = \frac{\max\{p_1, p_2, \ldots, p_n\}}{|a_{11}^{(0)}|},$$

where $p_1, p_2, \ldots, p_n$ are the pivots of A.

According to known theorems [10], it is clear that the stability of GE depends on the growth factor. If $g(n, A)$ is of order 1, not much growth has taken place, and the elimination process is stable. If $g(n, A)$ is bigger than this, we must expect instability. If GE is unstable, why is it so famous and so popular? The answer seems to be that although some matrices cause instability, these represent such an extraordinary small proportion of the set of all matrices that they "never" arise in practise simply for statistical reasons. This explanation gives rise to a statistical approach to the growth factor [11].

Cryer [2] defined $g(n) = \sup\{g(n, A) | A \in \mathbb{R}^{n\times n}, CP\}$. The problem of determining $g(n)$ for various values of n is called the *growth problem*. The determination of $g(n)$ in general remains a mystery. Wilkinson in [13] proved that $g(n) \leqslant [n\, 2\, 3^{1/2} \cdots n^{1/(n-1)}]^{1/2} \sim cn^{1/2} n^{\frac{1}{4}\log n}$ and that this bound is not attainable.

In [2] Cryer conjectured that "for real matrices $g(n, A) \leqslant n$, with equality if and only if A is a Hadamard matrix". This conjecture became one of the most famous open problems in Numerical Analysis and has been investigated by many mathematicians. It was finally shown to be false in 1991, however its second part is still an open problem.

It can be proved [3] that the magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = \frac{W(j)}{W(j-1)}, \quad j = 1, 2, \ldots, n, \ W(0) = 1. \tag{19}$$

So, it is obvious that the calculation of minors is important in order to study pivot structures, and moreover the growth problem for CP weighing matrices.

5.2. Specification of pivots of $W(n, n-k)$

We consider that the pivots in this section are derived from matrices with CP structure.

In [8] was proved the following Lemma 5:

Lemma 5. *Let W be a CP skew or symmetric matrix, of order $n \geqslant 6$, then if GE is performed on W the first two pivots are* 1 *and* 2.

Remark 8. Absolutely similar is the proof for a $W(n, n-k)$, so we can conclude that the first two pivots are 1 and 2.

Theorem 2. *When Gaussian Elimination is applied on a CP $W(n, n-k)$ the last two pivots are* (*in backward order*) $n-k$ *and* $\frac{n-k}{2}$.

Proof. It follows by applying the results of Propositions 1 and 2 (taking into consideration that the maximum minors of a CP matrix appear in its upper left corner) and Eq. (4) in formula (19). □

5.3. Specification of pivot patterns of $W(n, n-2)$

Proposition 9. *Let W be a CP $W(n, n-2)$, of order $n \geqslant 6$, then if GE is performed on W the third pivot is* 2.

Proof. It follows from Corollary 2 and Eq. (19) (taking into consideration that the upper left 2×2 corner of a CP $W(n, n-k)$ can be $[\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}]$). □

Proposition 10. *Let W be a CP $W(n, n-2)$, of order $n \geqslant 10$, then if GE is performed on W the fourth pivot is* 3 *or* 4 *or* $\frac{5}{2}$.

Proof. It follows from Corollary 2 and Eq. (19). □

Theorem 3. *When GE is applied on a CP $W(n, n-2)$ the last three pivots are* (*in backward order*) $n-2$, $\frac{n-2}{2}$ *and* $\frac{n-2}{2}$.

Proof. The last two values are given in Theorem 2. The third value from the end follows from Propositions 2 and 3 (taking into consideration that the maximum minors of a CP matrix appear in its upper left corner) and Eq. (19). □

Lemma 6. *If GE with complete pivoting is applied on a $W(6, 4)$ the pivot pattern is* $(1, 2, 2, 4, 4, 6)$.

Proof. It follows from Remark 8, Proposition 9 and Theorem 3. □

Lemma 7. *If GE with complete pivoting is applied on a $W(8, 6)$ the pivot patterns are* $\left(1, 2, 2, 4, \frac{3}{2}, 3, 3, 6\right)$ *or* $\left(1, 2, 2, \frac{5}{2}, \frac{12}{5}, 3, 3, 6\right)$.

Proof. From Remark 8, Proposition 9 and Theorem 3 we have the values for the first three and the last three pivots.

From Proposition 6 in combination with $W(3) = 4$ for a $W(n, n-2)$ with $n \geqslant 6$ (Corollary 2) and Eq. (19) we get

$$p_4 = \frac{W(4)}{W(3)} \Rightarrow p_4 = \frac{16}{4} \text{ or } \frac{10}{4} \Rightarrow p_4 = 4 \text{ or } \frac{5}{2}.$$

The fifth pivot will be calculated with help of the property that the product of the pivots equals the determinant of a matrix.

$$p_5 = \frac{\det(W(8,6))}{\prod_{i=1_{i\neq 5}}^{8} p_i} = \frac{6^4}{1\cdot 2\cdot 2\cdot 4\cdot 3\cdot 3\cdot 6} \quad \text{or} \quad \frac{6^4}{1\cdot 2\cdot 2\cdot \frac{5}{2}\cdot 3\cdot 3\cdot 6} \Rightarrow p_5 = \frac{3}{2} \quad \text{or} \quad \frac{12}{5}. \qquad \square$$

Lemma 8. *If GE with complete pivoting is applied on a* $W(10,8)$ *the pivot patterns are*

	Pivot patterns of $W(10,8)$
1	$(1,2,2,\frac{5}{2},\frac{16}{5},4,2,4,4,8)$ or
2	$(1,2,2,\frac{5}{2},2,\frac{24}{7},2,4,4,8)$ or
3	$(1,2,2,\frac{5}{2},2,4,2,4,4,8)$ or
4	$(1,2,2,\frac{5}{2},\frac{8}{3},4,2,4,4,8)$ or
5	$(1,2,2,3,\frac{8}{3},4,2,4,4,8)$ or
6	$(1,2,2,3,\frac{14}{5},4,2,4,4,8)$ or
7	$(1,2,2,4,2,4,2,4,4,8)$ or
8	$(1,2,2,4,\frac{14}{5},4,2,4,4,8)$ or
9	$(1,2,2,4,\frac{14}{5},4,\frac{8}{3},4,4,8)$ or
10	$(1,2,2,4,\frac{16}{5},4,\frac{8}{3},4,4,8)$

Proof. From Remark 8, Proposition 9 and Theorem 3 we have the values for the first three and the last three pivots.

From Corollary 2 and Eq. (19) we get

$$p_4 = \frac{W(4)}{W(3)} \Rightarrow p_4 = \frac{16}{4} \text{ or } \frac{12}{4} \text{ or } \frac{10}{4} \Rightarrow p_4 = 4 \text{ or } 3 \text{ or } \frac{5}{2}.$$

From Proposition 7 we have

$$W(5) = 32 \text{ or } 28 \text{ or } 20 \text{ for a } W(10,8).$$

The 5×5 matrices with determinant 32, which have been found previously with application of the algorithm, contain in the upper left corner the 4×4 matrices A_1, A_2 and A_3 with determinants 16, 12 and 10 respectively. The 5×5 matrix with determinant 28 contains in the upper left corner the 4×4 matrix A_3 with determinant 10. The 5×5 matrix with determinant 20 contains in the upper left corner the 4×4 matrix A_3 with determinant 10. So, the fifth pivot of $W(10,8)$ can be calculated using relationship (19):

$$p_5 = \frac{W(5)}{W(4)} \Rightarrow p_5 = \frac{32}{16} \text{ or } \frac{32}{12} \text{ or } \frac{32}{10} \text{ or } \frac{28}{10} \text{ or } \frac{20}{10} \Rightarrow p_5 = 2 \text{ or } \frac{8}{3} \text{ or } \frac{14}{5} \text{ or } \frac{16}{5}.$$

In the same manner, we go on to the sixth pivot: from Proposition 7 we have

$$W(6) = 128 \text{ or } 112 \text{ or } 96 \text{ or } 80 \text{ for a } W(10,8).$$

The 6×6 matrices with determinant 128 contain in the upper left corner the 5×5 matrix with determinant 32. The 6×6 matrices with determinant 112 contain in the upper left corner the 5×5 matrix with determinant 28. The 6×6 matrices with determinant 96 contain in the upper left corner the 5×5 matrix with determinant 28. The 6×6 matrices with determinant 80 contain in the upper left corner the 5×5 matrix with determinant 20. So, the sixth pivot of $W(10, 8)$ can be calculated using relationship (19):

$$p_6 = \frac{W(6)}{W(5)} \Rightarrow p_6 = \frac{128}{32} \text{ or } \frac{112}{28} \text{ or } \frac{96}{28} \text{ or } \frac{80}{20} \Rightarrow p_6 = 4 \text{ or } \frac{24}{7}.$$

The seventh pivot will be calculated with help of the property that the product of the pivots equals the determinant of a matrix. For the first case we have $p_1 = 1$, $p_2 = 2$, $p_3 = 2$, $p_4 = \frac{5}{2}$, $p_5 = \frac{16}{5}$, $p_6 = 4$, $p_8 = 4$, $p_9 = 4$, $p_{10} = 8$.

$$p_7 = \frac{\det(W(10, 8))}{\prod_{i=1_{i \neq 7}}^{10} p_i} = \frac{8^5}{1 \cdot 2 \cdot 2 \cdot \frac{5}{2} \cdot \frac{16}{5} \cdot 4 \cdot 4 \cdot 4 \cdot 8} \Rightarrow p_7 = 2.$$

Similarly we obtain the values of the seventh pivot for the other nine cases. □

Lemma 9. *If GE with complete pivoting is applied on a $W(12, 10)$ the pivot patterns are*

	Pivot patterns of $W(12, 10)$
1	$(1, 2, 2, \frac{5}{2}, \frac{5}{2}, 4, 3, 4, \frac{10}{3}, 5, 5, 10)$ or
2	$(1, 2, 2, \frac{5}{2}, \frac{5}{2}, 5, 2, 4, 4, 5, 5, 10)$ or
3	$(1, 2, 2, \frac{5}{2}, \frac{14}{5}, \frac{26}{7}, \frac{40}{13}, 5, \frac{5}{2}, 5, 5, 10)$ or
4	$(1, 2, 2, \frac{5}{2}, \frac{14}{5}, 4, \frac{45}{14}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
5	$(1, 2, 2, \frac{5}{2}, 3, \frac{10}{3}, 3, 4, \frac{10}{3}, 5, 5, 10)$ or
6	$(1, 2, 2, \frac{5}{2}, \frac{18}{5}, \frac{28}{9}, \frac{45}{14}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
7	$(1, 2, 2, \frac{5}{2}, \frac{18}{5}, \frac{65}{18}, \frac{36}{13}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
8	$(1, 2, 2, 3, \frac{5}{2}, \frac{10}{3}, 3, 4, \frac{10}{3}, 5, 5, 10)$ or
9	$(1, 2, 2, 3, \frac{5}{2}, 4, 3, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
10	$(1, 2, 2, 3, \frac{8}{3}, \frac{13}{4}, \frac{40}{13}, \frac{28}{9}, \frac{15}{4}, \frac{10}{3}, 5, 5, 10)$ or
11	$(1, 2, 2, 3, \frac{8}{3}, \frac{13}{4}, \frac{40}{13}, 5, \frac{5}{2}, 5, 5, 10)$ or
12	$(1, 2, 2, 3, \frac{8}{3}, \frac{7}{2}, \frac{45}{14}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
13	$(1, 2, 2, 3, \frac{8}{3}, 4, \frac{7}{2}, \frac{15}{4}, \frac{10}{3}, 5, 5, 10)$ or
14	$(1, 2, 2, 3, \frac{8}{3}, 4, \frac{7}{2}, 5, \frac{5}{2}, 5, 5, 10)$ or
15	$(1, 2, 2, 3, 3, \frac{28}{9}, \frac{45}{14}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
16	$(1, 2, 2, 3, 3, \frac{65}{18}, \frac{36}{13}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
17	$(1, 2, 2, 3, 3, 4, \frac{10}{3}, \frac{10}{3}, \frac{5}{2}, 5, 5, 10)$ or
18	$(1, 2, 2, 4, 2, \frac{7}{2}, \frac{45}{14}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
19	$(1, 2, 2, 4, 2, 4, \frac{5}{2}, \frac{15}{4}, \frac{10}{3}, 5, 5, 10)$ or
20	$(1, 2, 2, 4, 2, 4, \frac{5}{2}, 5, \frac{5}{2}, 5, 5, 10)$ or
21	$(1, 2, 2, 4, \frac{9}{4}, \frac{28}{9}, \frac{45}{14}, \frac{40}{9}, \frac{5}{2}, 5, 5, 10)$ or
22	$(1, 2, 2, 4, 3, \frac{10}{3}, 3, \frac{10}{3}, \frac{5}{2}, 5, 5, 10)$

Table 5

Pivot	n									
	6	8	10	12	18	20	22	28	34	38
p_1	1	1	1	1	1	1	1	1	1	1
p_2	2	2	2	2	2	2	2	2	2	2
p_3	2	2	2	2	2	2	2	2	2	2
p_4	–	$4, \frac{5}{2}$	$3, 4, \frac{5}{2}$	$3, 4, \frac{5}{2}$	$3, 4$	$3, 4$	$3, 4$	$3, 4$	$3, 4$	$3, 4$
p_5	–	$\frac{3}{2}, \frac{12}{5}$	$2, \frac{8}{3}, \frac{14}{5}$	$2, 3, \frac{9}{4}, \frac{5}{2},$	$2, \frac{5}{2}, 3,$	$2, \frac{5}{2}, 3,$	$\frac{5}{2}, 3, \frac{10}{3}$	$\frac{5}{2}, 3$	$2, 3, \frac{10}{3}, \frac{5}{2}$	$3, \frac{10}{3}$
	–		$\frac{16}{5}$	$\frac{8}{3}, \frac{14}{5}, \frac{18}{5}$	$\frac{10}{3}$	$\frac{10}{3}$				
p_6	–	–	$\frac{24}{7}, 4$	$4, 5, \frac{28}{9},$	$3, \frac{16}{5}, \frac{10}{3},$	$3, \frac{16}{5}, \frac{33}{10},$	$3, \frac{10}{3}, \frac{17}{5},$	$3, \frac{16}{5}, \frac{10}{3}, \frac{17}{5},$	$\frac{10}{3}, \frac{18}{5}, 4$	$\frac{10}{3}, \frac{18}{5}$
	–	–		$\frac{13}{4}, \frac{10}{3}, \frac{7}{2},$	$\frac{17}{5}, \frac{18}{5}, 4$	$\frac{10}{3}, \frac{17}{5}, \frac{32}{9},$	$\frac{18}{5}$	$\frac{18}{5}$		
	–	–		$\frac{65}{18}, \frac{26}{7}$		$\frac{18}{5}, \frac{11}{3}, \frac{34}{9}, 4$				
p_{n-5}	–	–	–	$2, \frac{5}{2}, \frac{36}{13}, 3$	$4, \frac{9}{2}, \frac{24}{5},$	$\frac{9}{2}, \frac{324}{65}, \frac{33}{10}, 5$	$5, 6, \frac{20}{3},$	$\frac{13}{12}, \frac{117}{16}, \frac{130}{17},$	$8, 10, 12,$	$9, \frac{81}{8}, \frac{180}{17},$
	–	–	–	$\frac{40}{13}, \frac{45}{14}, \frac{10}{3}$	$5, \frac{16}{3}, 6,$	$\frac{81}{16}, \frac{90}{17}, \frac{27}{5},$	$\frac{15}{2}, \frac{25}{4},$	$\frac{39}{5}, \frac{260}{33}, \frac{65}{8},$	$\frac{32}{3}, \frac{48}{5}, \frac{160}{17}$	$\frac{54}{5}, \frac{120}{11},$
	–	–	–		$\frac{50}{7}$	$\frac{60}{11}, \frac{45}{8}, 6,$		$\frac{117}{14}, \frac{26}{3}, \frac{39}{4}$		$\frac{45}{4}, 12, \frac{27}{2}$
	–	–	–			$\frac{27}{4}, \frac{36}{5}$				
p_{n-4}	–	–	–	$\frac{10}{3}, \frac{15}{4}, 4$	$\frac{16}{3}, 6, \frac{32}{5}$	$6, \frac{27}{4}, \frac{36}{5},$	$\frac{20}{3}, 8, 10$	$\frac{26}{3}, \frac{39}{4}, \frac{52}{5},$	$\frac{64}{5}, 16, \frac{32}{3}$	$12, \frac{72}{5}, 16,$
	–	–	–	$\frac{40}{9}, 5$	$8, \frac{64}{9}$	$8, 9, \frac{54}{7}$		$\frac{104}{9}, 13$		18
p_{n-3}	–	–	$2, \frac{8}{3}$	$\frac{5}{2}, \frac{10}{3}, 4$	$4, \frac{16}{3}$	$\frac{9}{2}, 6$	5	$\frac{13}{2}, \frac{26}{3}$	8	9
p_{n-2}	2	3	4	5	8	9	10	13	16	18
p_{n-1}	2	3	4	5	8	9	10	13	16	18
p_n	4	6	8	10	16	18	20	26	32	36
Growth	4	6	8	10	16	18	20	26	32	36

Proof. From Remark 8 and Propositions 9 and 10 we have the values for the first 4 pivots. From Theorem 3 we have the values for the three last pivots. The fifth, sixth and seventh pivot are found by applying a similar procedure combining Algorithm Extend and Proposition 8, similarly to the previous proof. In the same sense we could continue with the eighth pivot, but we wanted to use Algorithm Minors in order to avoid the high complexity of Algorithm Extend. We have seen with Algorithm Minors that $W(9) = 4000$ for a $W(12, 10)$ containing in the upper left 3×3 corner the matrix B_1, and so is the case for B_2. Similarly we can find with Algorithm Minors, if we use as 4×4 upper left corner the matrices A_1, A_2 and A_3, that $W(8) = 1600$ or 1200 or 1000 for the $W(12, 10)$. So, with use of (19) in combination with Proposition 8 we calculate all possible values for the eighth and ninth pivot. Alternatively, we could find the ninth pivot from the product of the pivots. □

Theorem 4. *The growth factors of the* $W(6, 4)$, $W(8, 6)$, $W(10, 8)$ *and* $W(12, 10)$ *are* 4, 6, 8 *and* 10 *respectively*.

Proof. The result follows from Lemmas 6–9 and from the definition for the growth factor given in Section 5.1. □

6. Experimental results

We studied, by computer, the pivots and growth factors for $W(n, n-2)$, $n = 6, 8, 10, 12, 18, 20, 22, 28, 34$ and 38 constructed by two or four circulant matrices [5] and obtained the results in Table 5. This table gives us all possible appearing values of the first six and last six pivots calculated by computer for the first few $W(n, n-2)$. For each value of n were tested 50,000–1,000,000 H-equivalent matrices and the corresponding pivot patterns were found with application of GE with complete pivoting. Similarly, we created Tables 7 and 9 for $W(n, n-3)$ and $W(n, n-4)$, respectively.

Interesting results in the size of pivots appear when GE is applied on CP weighing matrices of order n and weight $n-k$. These results are presented in the tables below and give rise to a new conjecture that can be posed for this category of matrices. In [8] has been studied the growth problem for CP skew and symmetric conference matrices. In these matrices, the growth is also large, and experimentally, we have been led to believe it equals $n-1$ and that special structure appears for the first few and last few pivots.

Table 6

Pivot	Values
p_1	1
p_2	2
p_3	2
p_4	$\frac{5}{2}$,3,4
p_5	$2, \frac{9}{4}, \frac{12}{5}, \frac{5}{2}, \frac{8}{3}, \frac{14}{5}, 3, \frac{16}{5}, \frac{10}{3}, \frac{18}{5}$
p_{n-4}	$\frac{n-2}{3}, \frac{3(n-2)}{8}, \frac{3(n-2)}{7}, \frac{2(n-2)}{5}, \frac{4(n-2)}{9}, \frac{n-2}{2}$
p_{n-3}	$\frac{n-2}{4}, \frac{n-2}{3}, \frac{n}{3}$
p_{n-2}	$\frac{n-2}{2}$
p_{n-1}	$\frac{n-2}{2}$
p_n	$n-2$

Table 7

Pivot	n			
	8	12	16	20
p_1	1	1	1	1
p_2	2	2	2	2
p_3	$\frac{3}{2}, 2$	2	2	2
p_4	3, 4	$\frac{5}{2}, \frac{9}{4}, 3, 4$	$\frac{5}{2}, 3, 4$	3, 4
p_5	–	$\frac{5}{2}, \frac{9}{4}, 3,$	$\frac{7}{4}, 2, \frac{9}{4}, \frac{7}{3}, \frac{5}{2}, \frac{8}{3},$	$2, \frac{9}{4}, \frac{5}{2},$
	–	$\frac{13}{5}, \frac{8}{3}, \frac{27}{10},$	$\frac{14}{5}, 3, \frac{16}{5}, \frac{10}{3}, \frac{18}{5}$	$3, \frac{10}{3}$
	–	$\frac{14}{5}, 3, \frac{16}{5}, \frac{10}{3}, \frac{18}{5}$		
p_6	–	$\frac{13}{5}, 3, \frac{13}{4}, \frac{10}{3},$	$\frac{5}{2}, \frac{8}{3}, \frac{27}{10}, \frac{49}{18}, \frac{14}{5}, \frac{17}{6}, \frac{26}{9}, \frac{29}{10}, \frac{35}{12},$	$\frac{5}{2}, \frac{8}{3}, \frac{14}{5},$
	–	$\frac{17}{5}, \frac{26}{7},$	$\frac{29}{9}, \frac{13}{4}, \frac{49}{15}, \frac{33}{10}, \frac{53}{16}, \frac{10}{3}, \frac{51}{16}, \frac{16}{5},$	$\frac{17}{6}, \frac{29}{10}, \frac{35}{12}, 3,$
	–	4, 5	$\frac{27}{8}, \frac{17}{5}, \frac{7}{2}, \frac{32}{9}, \frac{49}{16}, \frac{28}{9}, \frac{25}{8},$	$\frac{28}{9}, \frac{19}{6}, \frac{16}{5}, \frac{33}{10}, \frac{10}{3}, \frac{17}{5},$
	–		$\frac{53}{18}, 3, \frac{55}{18}, \frac{65}{18}, \frac{29}{8}, \frac{11}{3}, \frac{15}{4}, \frac{34}{9}, 4$	$\frac{32}{9}, \frac{18}{5}, \frac{65}{18}, \frac{11}{3}, \frac{15}{4}, \frac{34}{9}, 4$
p_{n-5}	–	$\frac{9}{5}, \frac{9}{4}, \frac{36}{13},$	$\frac{78}{25}, \frac{13}{4}, \frac{18}{5}, \frac{65}{18}, \frac{65}{12}$	$\frac{102}{25}, \frac{17}{4}, \frac{68}{15}$
	–	$3, \frac{16}{5}, \frac{18}{5},$	$\frac{26}{7}, \frac{65}{17}, \frac{39}{10}, \frac{208}{53}, \frac{26}{5}$	$\frac{153}{32}, 5, \frac{51}{10}, \frac{170}{33}, \frac{85}{16}, \frac{153}{26}$
	–	$\frac{15}{4}$	$\frac{130}{33}, \frac{65}{16}, \frac{104}{25}, \frac{117}{28}, \frac{39}{8}$	$\frac{136}{25}, \frac{17}{3}, \frac{51}{8}, \frac{34}{5}, \frac{85}{14}, \frac{51}{7}$
	–		$\frac{208}{49}, \frac{234}{55}, \frac{13}{3}, \frac{130}{29}, \frac{65}{14}, \frac{234}{49}$	$\frac{306}{65}, \frac{34}{7}, \frac{153}{28}, \frac{306}{55}, \frac{204}{35}, \frac{170}{29}$
p_{n-4}	–	$\frac{18}{5}, \frac{27}{8}, 4$	$\frac{13}{3}, \frac{39}{8}, \frac{26}{5}, \frac{39}{7}, \frac{52}{9}, \frac{13}{2}, \frac{52}{7}$	$\frac{17}{3}, \frac{51}{8}, \frac{34}{5},$
	–	$\frac{9}{2}$		$\frac{68}{9}, \frac{17}{2}, \frac{68}{7}$
p_{n-3}	$\frac{5}{4}, \frac{5}{3}$	$\frac{9}{4}, 3, \frac{18}{5}$	$\frac{13}{4}, \frac{13}{3}$	$\frac{17}{4}, \frac{17}{3}$
p_{n-2}	$\frac{5}{2}, \frac{10}{3}$	$\frac{9}{2}$	$\frac{13}{2}$	$\frac{17}{2}$
p_{n-1}	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{13}{2}$	$\frac{17}{2}$
p_n	5	9	13	17
Growth	5	9	13	17

Table 8

Pivot	Values
p_1	1
p_2	2
p_3	2
p_4	$\frac{5}{2}, \frac{9}{4}, 3, 4$
p_5	$\frac{7}{4}, 2, \frac{7}{3}, \frac{5}{2}, \frac{9}{4}, 3, \frac{13}{5}, \frac{8}{3}, \frac{27}{10}, \frac{14}{5}, 3, \frac{16}{5}, \frac{10}{3}, \frac{18}{5}$
p_{n-4}	$\frac{n-3}{3}, \frac{3(n-3)}{8}, \frac{3(n-3)}{7}, \frac{4(n-3)}{7}, \frac{2(n-3)}{5}, \frac{4(n-3)}{9}, \frac{n-3}{2}$
p_{n-3}	$\frac{n-3}{4}, \frac{n-3}{3}, \frac{3n}{10}$
p_{n-2}	$\frac{n-3}{2}$
p_{n-1}	$\frac{n-3}{2}$
p_n	$n-3$

Table 9

Pivot	n			
	8	12	16	20
p_1	1	1	1	1
p_2	2	2	2	2
p_3	$\frac{3}{2}$	2	2	2
p_4	3	$2, \frac{9}{4}, \frac{5}{2}, 3, 4$	$\frac{5}{2}, 3, 4$	$\frac{5}{2}, 3, 4$
p_5	–	$2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4},$	$2, \frac{9}{4}, \frac{7}{3}, \frac{5}{2}, \frac{13}{5},$	$2, \frac{9}{4}, \frac{5}{2}, \frac{8}{3}, 3, \frac{16}{5}, \frac{10}{3}, \frac{18}{5}$
	–	$3, \frac{16}{5}, 4$	$\frac{8}{3}, \frac{27}{10}, \frac{14}{5}, 3, \frac{16}{5}, \frac{10}{3}, \frac{18}{5}$	
p_6	–	$\frac{7}{3}, \frac{13}{5}, \frac{8}{3}, 3,$	$\frac{13}{5}, \frac{21}{8}, \frac{8}{3}, \frac{49}{18}, \frac{41}{15}, \frac{11}{4}, \frac{25}{9}, \frac{14}{5}, \frac{45}{16}, \frac{17}{6},$	$\frac{8}{3}, \frac{14}{5}, \frac{51}{18}, \frac{26}{9}, \frac{29}{10}, \frac{35}{12}, \frac{53}{18}, 3,$
	–	$\frac{16}{5}, \frac{36}{11}, \frac{17}{5},$	$\frac{43}{15}, \frac{23}{8}, \frac{29}{10}, \frac{35}{12}, \frac{44}{15}, \frac{47}{16}, \frac{53}{18}, 3, \frac{55}{18},$	$\frac{55}{18}, \frac{28}{9}, \frac{25}{8}, \frac{51}{16}, \frac{16}{5}, \frac{29}{9}, \frac{13}{4}, \frac{33}{10}, \frac{10}{3},$
	–	4	$\frac{49}{16}, \frac{46}{15}, \frac{31}{10}, \frac{28}{9}, \frac{25}{8}, \frac{141}{45}, \frac{19}{6}, \frac{51}{16}, \frac{16}{5},$	$\frac{27}{8}, \frac{17}{5}, \frac{55}{16}, \frac{7}{2}, \frac{32}{9}, \frac{18}{5}, \frac{65}{18}, \frac{29}{8}, \frac{11}{3},$
	–		$\frac{29}{9}, \frac{13}{4}, \frac{49}{15}, \frac{53}{16}, \frac{10}{3}, \frac{27}{8}, \frac{17}{5}, \frac{55}{16}, \frac{52}{15}, \frac{7}{2},$	$\frac{56}{15}, \frac{15}{4}, \frac{34}{9}, \frac{58}{15}, 4, \frac{25}{6}, \frac{64}{15}, \frac{13}{3}, \frac{22}{5}$
	–		$\frac{57}{16}, \frac{18}{5}, \frac{65}{18}, \frac{29}{8}, \frac{33}{9}, \frac{168}{45}, \frac{15}{4}, \frac{34}{9}, \frac{19}{5}, \frac{58}{15},$	
	–		$4, \frac{13}{3}, \frac{43}{14}, \frac{40}{13}, \frac{81}{26}, \frac{22}{7}, \frac{41}{13}, \frac{45}{14}$	
p_{n-5}	–	$2, \frac{8}{3}, 3, \frac{10}{3}$	$3, \frac{18}{5}, \frac{15}{4}, \frac{96}{25}, 4, \frac{64}{15}, \frac{108}{25},$	$\frac{96}{25}, 4\frac{64}{15}, \frac{288}{65}, \frac{40}{9}, \frac{9}{2}, \frac{32}{7}, \frac{80}{17}, \frac{24}{5},$
	–	$\frac{32}{9}, 4, \frac{44}{17}, \frac{44}{13}, \frac{24}{7}$	$\frac{48}{11}, \frac{40}{9}, \frac{9}{2}, \frac{24}{5}, \frac{54}{11}, 5,$	$\frac{160}{33}, 5, \frac{96}{19}, \frac{128}{25}, \frac{36}{7}, \frac{256}{49}, \frac{288}{55}, \frac{16}{3},$
	–		$\frac{648}{195}, \frac{24}{7}, \frac{60}{17}, \frac{192}{53}, \frac{48}{13}, \frac{72}{19}, \frac{27}{7}, \frac{168}{43},$	$\frac{192}{35}, \frac{160}{29}, \frac{72}{13}, \frac{40}{7}, \frac{480}{81}, 6, \frac{80}{13}, \frac{32}{5}, \frac{48}{7}$
	–		$\frac{192}{49}, \frac{216}{55}, \frac{216}{53}, \frac{192}{47}, \frac{120}{29}, \frac{96}{23}, \frac{180}{43},$	
	–		$\frac{72}{17}, \frac{30}{7}, \frac{216}{49}, \frac{32}{7}, \frac{216}{47}, \frac{60}{13}, \frac{216}{43}$	
p_{n-4}	–	$\frac{8}{3}, \frac{16}{5}, \frac{10}{3},$	$4, \frac{9}{2}, \frac{24}{5}, \frac{16}{3}, 6,$	$\frac{16}{3}, 6, \frac{32}{5}, \frac{64}{9}, 8, \frac{64}{7}$
	–	$\frac{40}{11}, 4, \frac{24}{5}, \frac{16}{3}$	$\frac{72}{14}, \frac{96}{14}$	
p_{n-3}	$\frac{4}{3}$	$2, \frac{8}{3}, \frac{16}{5}, 4$	3, 4	$4, \frac{16}{3}$
p_{n-2}	$\frac{8}{3}$	4	6	8
p_{n-1}	2	4	6	8
p_n	4	8	12	16
Growth	4	8	12	16

The growth conjecture for $W(n, n-k)$

Let W be a CP $W(n, n-k)$. Reduce W by GE. Then, for large enough n,

(i) $g(n, W) = n - k$.
(ii) The three last pivots are equal to $\frac{n-k}{2}, \frac{n-k}{2}, n-k$.
(iii) Every pivot before the last has magnitude at most $n - k$.
(iv) The first three pivots are equal to 1, 2, 2. The fourth pivot can take the values 3 or 4 or $\frac{5}{2}$.

In Table 6 we present all the values appearing for the first five and last five pivots after applying Gaussian Elimination with complete pivoting on $W(n, n-2)$ of order $n \geqslant 6$.

In Table 8 we present all the values appearing for the first five and last five pivots after applying Gaussian Elimination with complete pivoting on $W(n, n-3)$ of orders 12,16 and 20.

In Table 10 we present all the values appearing for the first five and last five pivots after applying Gaussian Elimination with complete pivoting on $W(n, n-4)$ of order 12,16 and 20.

Table 10

Pivot	Values
p_1	1
p_2	2
p_3	2
p_4	$2, \frac{9}{4}, \frac{5}{2}, 3, 4$
p_5	$2, \frac{9}{4}, \frac{7}{3}, \frac{5}{2}, \frac{13}{5}, \frac{8}{3}, \frac{27}{10}, \frac{11}{4}, \frac{14}{5}, 3, \frac{16}{5}, \frac{10}{3}, \frac{18}{5}, 4$
p_{n-4}	$\frac{n-4}{3}, \frac{3(n-4)}{8}, \frac{4(n-4)}{7}, \frac{2(n-4)}{5}, \frac{4(n-4)}{9}, \frac{n-4}{2}, \frac{5(n-4)}{12}, \frac{5(n-4)}{11}, \frac{3(n-4)}{5}, \frac{2(n-4)}{3}$
p_{n-3}	$\frac{n-4}{4}, \frac{n-4}{3}, \frac{n-4}{2}, \frac{2(n-4)}{5}$
p_{n-2}	$\frac{n-4}{2}$
p_{n-1}	$\frac{n-4}{2}$
p_n	$n-4$

7. Conclusions

The subject of our research is to find an efficient way for calculating the minors of weighing matrices $W(n, n-k)$. For achieving this purpose we provided three tools: direct (analytic) computations of minors, counting techniques for specifying the existence of certain submatrices inside $W(n, n-k)$ and the notions developed in Algorithm Minors.

The ideas presented in this work are valid for general n and can be used as the fundamental basis, on which the calculation of minors of weighing matrices of small orders, such as $W(14, 12)$ and $W(16, 14)$, can be developed. Algorithm Minors theoretically can proceed to the computation of any $n-j$ minor. Another benefit is that the proposed algorithms used for the purpose of this paper can be modified appropriately and adapted for the specification of minors of other weighing matrices $W(n, n-k)$, $k = 3, 4, \ldots$, and also for other orthogonal matrices. An issue under consideration is the demonstration of all possible values of minors of weighing matrices. Finally, the symbolic implementations from a software point of view of the constructive procedure of the proof of Proposition 3 and of the Counting Techniques of Section 3.1 are currently under research, too.

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Appendix. 7 × 7 Matrices attaining all possible determinant values

$$0 \begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & - & - \\ + & - & - & - & - & - & - \\ + & - & - & - & - & - & - \end{bmatrix} \qquad 64 \begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & - & - \\ + & - & - & - & - & - & + \\ + & - & - & - & - & + & - \end{bmatrix}$$

$$128\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & - & + \\ + & - & - & - & - & + & - \\ + & - & - & + & - & + & + \end{bmatrix} \quad 192\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & - & + \\ + & - & - & - & - & + & - \\ + & - & - & - & + & + & + \end{bmatrix}$$

$$256\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & - & + \\ + & - & - & - & - & + & - \\ + & - & - & - & + & - & + \end{bmatrix} \quad 320\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & - & + \\ + & - & - & - & - & + & - \\ + & - & - & - & + & - & - \end{bmatrix}$$

$$384\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & + & + \\ + & - & - & - & + & - & + \\ + & - & - & + & + & + & - \end{bmatrix} \quad 448\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & - \\ + & - & - & - & - & + & + \\ + & - & - & - & + & - & + \\ + & - & - & - & + & + & - \end{bmatrix}$$

$$512\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & - \\ + & - & + & - & - & - & + \\ + & - & - & - & - & + & + \\ + & - & - & - & + & - & + \\ + & - & + & - & + & + & - \end{bmatrix} \quad 576\begin{bmatrix} + & + & + & + & + & + & + \\ + & - & - & + & - & - & - \\ + & + & - & - & - & - & + \\ + & - & + & - & - & - & + \\ + & - & - & - & + & + & - \\ + & + & + & - & - & + & - \\ + & + & + & - & + & - & - \end{bmatrix}$$

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